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Variable structure control of a singular predator-prey system with the influence of toxic substances

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A B S T R A C T

In order to describe the fast-slow process such as a one-off discharge and bioconcentration on tissues and organs of toxic substances, a singular predator-prey model with toxic substances is proposed using the principles of biodynamics and ecotoxicology. And the stability conditions in ideal state are given. Then according to the principles of variable structure control, the variable structure controller is designed to make the populations persistent and the concentration of toxic substances small enough in the environment. Finally, some computer simulations are carried out to prove the results. The study provides not only the background for the singular system models, but also the theoretical basis for the management of toxic substances.

1. Introduction

With the development of industry and agriculture, the environmental pollution becomes more and more serious: the waste gas, waste water and waste residue with toxic substances in industry and the pesticide residues in agriculture. On the one hand, the toxic substances above are absorbed into animal bodies and not decomposed easily, they damage the tissues and organs by accumulating. On the other hand, the toxic substances have critical negative effects on eco-environment. It is necessary to study the effect of toxic substances on the populations and the eco-environment.

As toxic substances, heavy metals are pollutants from various natural process (including erosion and weathering of the bedrock) and human activities (such as waste water from factory, exhaust fumes from vehicle and the residues of pesticide), accounting for more than 90% of total content in the aquatic ecosystems, which would inevitably cause severe entrophication and heavy metals pollution [1–3]. Due to the inherent toxicity, persistence and non-degradability, the contamination of heavy metals has been of great concern, many pollution events are reported on a global scale [4–10]. From the above, it is seen that the study on the contamination of heavy metals mainly focuses on the investigation and analysis of practical problems. But the dynamics models are few and the corresponding control models are fewer.

In general, the discharge of toxic substances such as heavy metals may be divided into the two main ways: continuous and discontinuous. For the former, some ordinary differential equations [11] and partial differential equations [12] are used to describe the process; for the latter, some impulse differential equations [13,14] are often used to show the phenomenon. If the heavy metals are discharged into the river at one time, the system shows the fast process. While the hazards on human have a comparatively slow dynamics. A singular system model may be employed to describe the slow-fast process. A singular system model is coupled with differential equations and algebraic equations, in which differential equations and the algebraic equations are used to describe the slow

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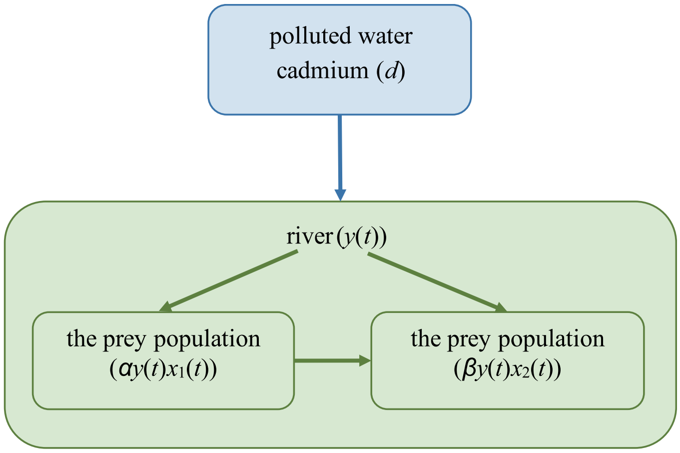

Fig. 2.1. Concentration distribution of cadmium.

and the fast process, respectively. The earlier work on the topic is reflected in the books [15,16]. Between 1990s and 2000s, singular system models are applied mainly in the fields of economics [17], chemical process [18], aerospace engineering [19] and power systems [20,21], etc. In the last decade, some singular system models are also applied to describe biological economic systems [22–25]. But less study on the model with heavy metals is focused on the singular systems, especially scarce in the description of fast-slow process.

With the development of control theory, more and more control methods are applied in some models with toxic substances, such as state feedback control [26] and adaptive control [27]. Compared with the control methods above, variable structure control has such advantages as simple algorithm, the higher reliability and the full self-adaption [28,29]. And it is often used to deal with some complex non-linear systems with uncertain parameters and external disturbances. Because the systems with toxic substances often show the above characteristics, the variable structure control is applied in the regulation of toxic substances in the paper, which provides the theoretical basis for the management of toxic substances.

From the above, a singular system model is proposed to describe the fast-slow process such as a one-off discharge of heavy metals and the hazards on human; and then the variable structure controller is designed to reduce the contamination level of heavy metals to the environmental quality standard.

2. Modeling

In the natural world, heavy metals are absorbed into the animal bodies by means of diet and exposure, and decomposed difficultly. For example, due to the half life of 10–30 years, cadmium does great harm to liver, kidney by accumulating. Considering the potential danger of heavy metals to the tissues and organs, it is necessary to analyze their influence on the populations.

First, suppose that there are the shrimp populations and the fish populations in a river, they are described as the predator-prey model [30]

$$\begin{cases} \dot{x}_1(t) = rx_1(t) - \mu_1 x_1(t)x_2(t), \\ \dot{x}_2(t) = \mu_{10}x_1(t)x_2(t) - \mu_{20}x_2(t), \end{cases} \tag{2.1}$$

where $x_1(t)$ is the density of the shrimp populations and $x_2(t)$ is the density of the fish populations at time t; in the absence of the fish populations, the shrimp populations grow unboundedly in a Malthusian way, which is described as the term $rx_1(t)$; the effect of the fish populations may reduce the shrimp populations, which is described as the term $-\mu_1 x_1(t)x_2(t)$; the contribution of shrimp populations to the fish populations is described as the term $\mu_{10}x_1(t)x_2(t)$; in the absence of the shrimp populations, the death rate of the fish populations is $-\mu_{20}$; r, μ_1, μ_{10} and μ_{20} are positive constants, $\mu_1 \geq \mu_{10}$.

According to the principles of biodynamics and ecotoxicology, the following assumptions are given.

Assumption I. The heavy metal such as cadmium is discharged into a river at one time, d is the total concentration of cadmium emitted into the river. If $y(t)$ is the concentration of cadmium at time t in the river, then

$$\dot{y}(t) = \lim_{t\to 0}\frac{d-0}{\Delta t} = \infty. \tag{2.2}$$

The formula Eq. (2.2) means the change rate of cadmium in the river increases dramatically in a short time, that is, the system reaches a new balance as quickly as possible.

Assumption II. Because the cadmium in the river can be absorbed by organisms and decomposed difficultly, with time going by, there is more cadmium in the shrimp populations and the fish populations than that in the river, namely the bioconcentration [31], which can be described as the bioconcentration factor BCF. In fact, the BCF is related to the concentration of toxic substances in a body and that in the environment, which is given

$$\mathrm{BCF} = \frac{\mathrm{A}}{\mathrm{D}}. \tag{2.3}$$

where A is the concentration of toxic substances in a body; D is the concentration of toxic substances in the environment. Denote BCF of the shrimp populations as α and that of the fish populations as β. According to the literature [31], BCF becomes larger with the increase of trophic level, that is $\beta > \alpha$(See Fig. 2.1). In Fig. 2.1, the term $\alpha y(t)x_1(t)$, $\beta y(t)x_2(t)$ are the total concentration of cadmium in the shrimp populations and the fish populations at time t, respectively.

Based on Assumptions I and II, the following model is employed to describe the fast process such as a one-off discharge

$$d = y(t) + \alpha y(t)x_1(t) + \beta y(t)x_2(t). \tag{2.4}$$

Assumption III. Since the bioconcentration of cadmium does harm to the shrimp populations and the fish populations. For the model Eq. (2.1), the influence rate of the cadmium on the shrimp populations is proportional to its concentration, and the corresponding factor is μ_2 ($\mu_2 > 0$). Similarly, the influence rate of the cadmium on the fish populations is proportional to its concentration, and the corresponding factor is μ_3 ($\mu_3 > 0$). According to the literature [31], BCF becomes larger with the increase of trophic level, that is $\mu_3 > \mu_2$. Thus, the process above is described by the following model

$$\begin{cases} \dot{x}_1(t) = rx_1(t) - \mu_1 x_1(t)x_2(t) - \mu_2 \alpha y(t)x_1(t), \\ \dot{x}_2(t) = \mu_{10}x_1(t)x_2(t) - \mu_{20}x_2(t) - \mu_3 \beta y(t)x_2(t). \end{cases} \tag{2.5}$$

Remark 2.1. Compared with the model Eq. (2.4), the model Eq. (2.5) shows the slow process of the system.

Combing the model Eq. (2.4) with the model Eq. (2.5), a singular predator-prey model with toxic substances is obtained

$$\begin{cases} \dot{x}_1(t) = rx_1(t) - \mu_1 x_1(t)x_2(t) - \mu_2 \alpha y(t)x_1(t), \\ \dot{x}_2(t) = \mu_{10}x_1(t)x_2(t) - \mu_{20}x_2(t) - \mu_3 \beta y(t)x_2(t), \\ 0 = y(t) + \alpha y(t)x_1(t) + \beta y(t)x_2(t) - d, \\ x_1(0) = x_{10} > 0, \\ x_2(0) = x_{20} > 0, \\ y(0) = y_0 > 0. \end{cases} \tag{2.6}$$

where x_{10}, x_{20}, y_0 are the corresponding initial values.

Remark 2.2. For the model Eq. (2.6), the cadmium in the river is absorbed in the shrimp populations and the fish populations. The cadmium accumulates in the tissues and organs of human body by consuming the fish populations and the shrimp populations with cadmium, which has a potential hazard to human being. It is necessary to reduce the cadmium in the shrimp populations, the fish populations and the river as quickly as possible by some control means.

In the following, the management model corresponding to the model Eq. (2.6) is proposed.

- **Control I** One part of the cadmium may be removed by harvesting on fish populations with cadmium, $u_1(t)$ is the harvesting rate for the fish populations, the following model is got

$$\dot{x}_2(t) = \mu_{10}x_1(t)x_2(t) - \mu_{20}x_2(t) - \mu_3 \beta y(t)x_2(t) - u_1(t). \tag{2.7}$$

- **Control II** The other part of the cadmium may be removed by means of chemical precipitation or microorganisms removal, $u_2(t)$ is the rate of control related to the concentration, the following model is got

$$0 = y(t) + \alpha y(t)x_1(t) + \beta y(t)x_2(t) - d - y(t)u_2(t). \tag{2.8}$$

According to Control I and Control II, the management model corresponding to the model Eq. (2.6) is obtained

$$\begin{cases} \dot{x}_1(t) = rx_1(t) - \mu_1 x_1(t)x_2(t) - \mu_2 \alpha y(t))x_2(t), \\ \dot{x}_2(t) = \mu_{10}x_1(t)x_2(t) - \mu_{20}x_2(t) - \mu_3 \beta y(t)x_2(t) - u_1(t), \\ 0 = y(t) + \alpha y(t)x_1(t) + \beta y(t)x_2(t) - d - y(t)u_2(t), \\ x_1(0) = x_{10} > 0, \\ x_2(0) = x_{20} > 0, \\ y(0) = y_0 > 0, \end{cases} \tag{2.9}$$

where r, μ_1, μ_2, μ_{10}, μ_{20}, μ_3, α, β, d are the same as the above.

For the models above, the following tasks are finished: For the model Eq. (2.6), the corresponding local stability is discussed using the stability theory of singular systems. For the model Eq. (2.9), the variable structure controller is designed to make the system stable at ideal state in finite time using the principles of variable structure control. Finally, some computer simulations are exerted to prove the results.

3. Model analysis

The model Eq. (2.6) is shortened as

$$\begin{cases} \dot{x}_1 = rx_1 - \mu_1 x_1 x_2 - \mu_2 \alpha x_2 y, \\ \dot{x}_2 = \mu_{10}x_1x_2 - \mu_{20}x_2 - \mu_3 \beta x_2 y, \\ 0 = y + \alpha x_1 y + \beta x_2 y - d, \end{cases} \tag{3.1}$$

where $x_1,\ x_2,\ \mu_1,\ \mu_2,\ \mu_{10},\ \mu_{20}, \alpha,\ \beta$ have the same meaning as those in the model Eq. (2.9)

The model Eq. (3.1) has the following equilibria,

$$P_1^*(0,0,d),\quad P_2^*\left(\frac{\mu_2}{r}d - \frac{1}{\alpha}, 0, \frac{r}{\mu_2\alpha}\right),\quad P_3^*\left(\frac{\mu_{20} + \mu_3\beta y^*3_1}{\mu_{10}}, \frac{r - \mu_2\alpha y_1^*}{\mu_1}, y_1^*\right)$$
$$Q_1^*\left(0, -\left(\frac{\mu_3}{\mu_2}d + \frac{1}{\beta}\right), -\frac{\mu_{20}}{\mu_3\beta}\right),\quad Q_2^*\left(\frac{\mu_{20} + \mu_3\beta y_2^*}{\mu_{10}}, \frac{r - \mu_2\alpha y_2^*}{\mu_1}, y_2^*\right).$$

where y_1^*, y_2^* are the roots of the equation

$$\left(\frac{\mu_3}{\mu_{10}} - \frac{\mu_2}{\mu_1}\right)\alpha\beta y^2 + \left(1 + \frac{\mu_{20}\alpha}{\mu_{10}} + \frac{r\beta}{\mu_1}\right)y - d = 0.$$

For the equilibria above, the following theorem is given.

Theorem 3.1. *For the model Eq. (3.1),*

(1) *the point P_1^* is a non-negative equilibrium;*
(2) *if $r < \mu_2\alpha d$, then the point P_2^* is a non-negative equilibrium;*
(3) *if $\frac{r-\mu_2\alpha y_1^*}{\mu_1} > 0$, then the point P_3^* is a non-negative equilibrium;*
(4) *the point Q_1^* is a negative equilibrium;*
(5) *if $\mu_3 > \mu_2$, then the point Q_2^* is a negative equilibrium;*
if $\left(\frac{\mu_3}{\mu_{10}} - \frac{\mu_2}{\mu_1}\right)\langle 0, \sqrt{\left(1 + \frac{\mu_{20}\alpha}{\mu_{10}} + \frac{r\beta}{\mu_1}\right)^2 + 4\left(\frac{\mu_3}{\mu_{10}} - \frac{\mu_2}{\mu_1}\right)\alpha\beta d} > 0$ and $\frac{r-\mu_2\alpha y_1^*}{\mu_1} > 0$, then the point Q_2^* is a non-negative equilibrium.

Remark 3.1. In Theorem 3.1, if $\mu_3 > \mu_2$, then Q_2^* is a negative equilibrium; And the point Q_1^* is also a negative equilibrium. From the view of ecological meaning, the two points are omitted.

For the model Eq. (3.1), let

$$\mathbf{F}(\mathbf{x}, y) = \begin{pmatrix} rx_1 - \mu_1 x_1 x_2 - \mu_2 \alpha x_1 y \\ \mu_{10}x_1x_2 - \mu_{20}x_2 - \mu_3\beta x_2 y \end{pmatrix},$$
$$G(\mathbf{x}, y) = y - \alpha y x_1 - \beta y x_2 - d,$$

where $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$.

In the following, the local stability of the model Eq. (3.1) at the point P_1^*, the point P_2^* and the point P_3^* is discussed, respectively.

3.1. The equilibrium P_1^*

For the equilibrium P_1^*, the following theorem is given.

Theorem 3.2. *For the model Eq. (3.1),*

(a) *If $r < \mu_2\alpha d$, then the equilibrium P_1^* is local stable;*
(b) *If $r > \mu_2\alpha d$, then the equilibrium P_1^* is unstable.*

Proof. Because

$$\det \mathrm{D_y} G(\mathbf{x}, y)|_{P_1^*} = 1,$$

Jacobian matrix of the model Eq. (3.1) at the equilibrium P_1^* is

$$\begin{aligned} \mathbf{J}|_{P_1^*} &= \left(\mathrm{D_x}\mathbf{F}(\mathbf{x}, y) - \mathrm{D_y}\mathbf{F}(\mathbf{x}, y)\left(\mathrm{D_y}G(\mathbf{x}, y)\right)^{-1}\mathrm{D_x}G(\mathbf{x}, y)\right)|_{P_1^*} \\ &= \begin{pmatrix} r - \mu_2\alpha d & 0 \\ 0 & -\mu_{20} - \mu_3\beta d \end{pmatrix}. \end{aligned}$$

The corresponding characteristic equation is

$$(\lambda - (r - \mu_2\alpha d))(\lambda - (-\mu_{20} - \mu_3\beta d)) = 0, \tag{3.2}$$

and the characteristic roots are

$$\lambda_1 = r - \mu_2\alpha d, \quad \lambda_2 = -\mu_{20} - \mu_3\beta d.$$

According to the literature [32], if $r < \mu_2\alpha d$, then the equilibrium P_1^* is locally stable; if $r > \mu_2\alpha d$, then the equilibrium P_1^* is unstable.□

3.2. The equilibrium P_2^*

For the equilibrium P_2^*, the following theorem is given.

Theorem 3.3. *For the model Eq. (3.1),*

(a) *If $\mu_2\alpha d < r$, then the equilibrium P_2^* is locally stable;*
(b) *If $\mu_2\alpha d > r$ or $\frac{\mu_{10}}{r\alpha}(\mu_2\alpha d - r) - \mu_{20} - \frac{\mu_3\beta r}{\mu_2\alpha} > 0$, then the equilibrium P_2^* is unstable.*

Proof. Similar to the proof of Theorem 3.2, the characteristic equation of $\mathbf{J}|_{P_2^*}$ is

$$\left(\lambda - \frac{r}{\mu_2\alpha d}(\mu_2\alpha d - r)\right)\left(\lambda - \left(\frac{\mu_{10}}{r\alpha}(\mu_2\alpha d - r) - \mu_{20} - \frac{\mu_3\beta r}{\mu_2\alpha}\right)\right) = 0. \tag{3.3}$$

The corresponding characteristic roots are

$$\lambda_1 = \frac{r}{\mu_2\alpha d}(\mu_2\alpha d - r), \quad \lambda_2 = \frac{\mu_{10}}{r\alpha}(\mu_2\alpha d - r) - \mu_{20} - \frac{\mu_3\beta r}{\mu_2\alpha}.$$

According to the literature [32], if $\mu_2\alpha d < r$, then the equilibrium P_2^* is locally stable; if $\mu_2\alpha d > r$ or $\frac{\mu_{10}}{r\alpha}(\mu_2\alpha d - r) - \mu_{20} - \frac{\mu_3\beta r}{\mu_2\alpha} > 0$, then the equilibrium P_2^* is unstable.□

3.3. The equilibrium P_3^*

For the equilibrium P_3^*, it is hard to study the corresponding local stability by means of the same analysis as the point P_1^* and P_2^*, thus the point P_3^* is studied using theorems in the literature [33].

According to the literature [33], the following lemmas are given.

Lemma 3.1. *If there exists a transformation*

$$\chi = \mathbf{Q}\overline{\chi}, \tag{3.4}$$

then the model Eq. (3.1) is changed into

$$\begin{cases} \dot{\overline{x}}_1 = \overline{F}_1(\overline{\boldsymbol{\chi}}), \\ \dot{\overline{x}}_2 = \overline{F}_2(\overline{\boldsymbol{\chi}}), \\ 0 = \overline{G}(\overline{\boldsymbol{\chi}}), \end{cases} \tag{3.5}$$

where

$$\boldsymbol{\chi} = (x_1, x_2, y)^T, \quad \overline{\boldsymbol{\chi}} = (\overline{x}_1, \overline{x}_2, \overline{y})^T,$$

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{\alpha y^*}{1+\alpha x_1^*+\beta x_2^*} & -\dfrac{\beta y^*}{1+\alpha x_1^*+\beta x_2^*} & 1 \end{pmatrix},$$

$$x_1^* = \frac{\mu_{20}+\mu_3\beta y^*}{\mu_{10}}, \quad x_2^* = \frac{r-\mu_2\alpha y^*}{\mu_1}, \quad M = \frac{y^*}{1+\alpha x_1^*+\beta x_2^*},$$

$$\overline{F}_1(\overline{\boldsymbol{\chi}}) = r\overline{x}_1 + \mu_2\alpha^2 M\overline{x}_1^2 - (\mu_1 - \mu_2\alpha\beta M)\overline{x}_1\overline{x}_2 - \mu_2\alpha\overline{x}_1\overline{y},$$
$$\overline{F}_2(\overline{\boldsymbol{\chi}}) = -\mu_{20}\overline{x}_2 + \mu_3\beta^2 M\overline{x}_2^2 + (\mu_{10}+\mu_3\alpha\beta M)\overline{x}_1\overline{x}_2 - \mu_3\beta\overline{x}_2\overline{y}, \mathrm{D}_{\overline{\mathbf{x}}}\overline{G}(\overline{\boldsymbol{\chi}})\dot{\overline{\mathbf{x}}}$$
$$\overline{G}(\overline{\boldsymbol{\chi}}) = -(\mathrm{D}_{\overline{y}}\overline{G}(\overline{\boldsymbol{\chi}}))^{-1}$$
$$= \big(-\alpha\overline{x}_1 - \beta\overline{x}_2 - \alpha^2\overline{x}_1^2 - \beta^2\overline{x}_2^2 - 2\alpha\beta\overline{x}_1\overline{x}_2\big)M + \overline{y} + \alpha\overline{x}_1\overline{y} + \beta\overline{x}_2\overline{y} - d,$$

$$\big(DG(\overline{\boldsymbol{\chi}})|_{p_3^*}\mathbf{Q} = \big(0, 0, 1+\alpha x_1^*+\beta x_2^*\big).$$

Lemma 3.2. *For the model Eq. (3.5), the corresponding parameterized model is*

$$\dot{\boldsymbol{\xi}} = \mathbf{E}\boldsymbol{\xi} + o(\boldsymbol{\xi}), \tag{3.6}$$

where

$$\boldsymbol{\xi} = (\xi_1, \xi_2)^{\mathrm{T}}, \quad \mathbf{E} = \mathbf{U}_{\mathbf{0}}^{\mathrm{T}}\left(\mathrm{D}\overline{\mathbf{F}}(\overline{x})_{|_{p_3^*}}\right)\begin{pmatrix} \mathrm{D}\overline{G}(\overline{x}) \\ \mathbf{U}_{\mathbf{0}}^{\mathrm{T}} \end{pmatrix}^{-1}_{|_{p_3^*}}\begin{pmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{pmatrix}.$$

$$\overline{\boldsymbol{\chi}} = \psi(\boldsymbol{\xi}) = \overline{\boldsymbol{\chi}}^* + \mathbf{U}_0\overline{\boldsymbol{\chi}} + \mathbf{V}_0 h(\overline{\boldsymbol{\chi}}), \quad \overline{G}(\overline{\boldsymbol{\chi}}) = 0,$$

$$\overline{\boldsymbol{\chi}}^* = \left(\overline{x}_1^*, \overline{x}_2^*, \overline{y}^*\right)^T, \mathbf{U}_{\mathbf{0}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{V}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and $h(\overline{\boldsymbol{\chi}})$: $R^2 \to R^1$ is a smooth mapping.

Proof. The Jacobian matrix $\mathrm{D}\psi(\boldsymbol{\xi})$ of the local parameterized function of the model Eq. (3.6) has size 3×2.

Differentiate $\overline{G}(\psi(\boldsymbol{\xi})) = 0$, the following is got

$$\mathrm{D}\overline{G}(\overline{\boldsymbol{\chi}})\mathrm{D}\psi(\boldsymbol{\xi}) = 0. \tag{3.7}$$

Differentiate the model Eq. (3.6) and premultiply both sides of the equation by $\mathbf{U}_{\mathbf{0}}^{\mathrm{T}}$, the following is got

$$\mathbf{U}_{\mathbf{0}}^{\mathrm{T}}\mathrm{D}\psi(\boldsymbol{\xi}) = \mathbf{I}_2, \tag{3.8}$$

where $\mathbf{I}_2$ is an identity matrix.

According to the literature [28], because $\det(\mathrm{D}\overline{G}(\overline{\boldsymbol{\chi}})\mathbf{V}_0) = 1 + \alpha x_1 + \beta x_2 \neq 0$, thus $\begin{pmatrix} \mathrm{D}\overline{G}(\overline{\boldsymbol{\chi}}) \\ \mathbf{U}_{\mathbf{0}}^{\mathrm{T}} \end{pmatrix}$ is invertible.

Combine the model Eq. (3.5) with the formula Eq. (3.7), the following is got

$$\mathrm{D}\psi(\boldsymbol{\xi}) = \begin{pmatrix} \mathrm{D}\overline{G}(\overline{\boldsymbol{\chi}}) \\ \mathbf{U}_{\mathbf{0}}^{\mathrm{T}} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{pmatrix}. \tag{3.9}$$

Substitute $\overline{\boldsymbol{\chi}} = \psi(\boldsymbol{\xi})$ into the model Eq. (3.5), the following is got

$$D\psi(\xi)\dot{\xi} = \overline{\mathbf{F}}(\psi(\xi)). \tag{3.10}$$

Substitute the formula Eq. (3.9) into the formula Eq. (3.10), the following is got

$$\begin{pmatrix} D\overline{G}(\overline{\chi}) \\ \mathbf{U}_0^T \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{pmatrix} \dot{\xi} = \overline{\mathbf{F}}(\psi(\xi)), \tag{3.11}$$

Thus

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{pmatrix} \dot{\xi} = \begin{pmatrix} D\overline{G}(\overline{\chi})\overline{\mathbf{F}}(\psi(\xi)) \\ \mathbf{U}_0^T\overline{\mathbf{F}}(\psi(\xi)) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{U}_0^T\overline{\mathbf{F}}(\psi(\xi)) \end{pmatrix}, \tag{3.12}$$

the equivalent form of the model Eq. (3.12) is

$$\dot{\xi} = \mathbf{U}_0^T\overline{\mathbf{F}}(\psi(\xi)). \tag{3.13}$$

The Taylor expansion of the model Eq. (3.6) at the equilibrium P_3^* is

$$\dot{\xi} = \mathbf{E}\xi + o(\xi), \tag{3.14}$$

where

$$\mathbf{E} = \mathbf{U}_0^T \left(D\overline{\mathbf{F}}(\overline{\chi})_{|_{P_3^*}} \right) \begin{pmatrix} D\overline{G}(\overline{\chi}) \\ \mathbf{U}_0^T \end{pmatrix}^{-1}_{|_{P_3^*}} \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{pmatrix}.$$

□ Based on Lemmas 3.1 and 3.2, the following theorem is given.

Theorem 3.4. *For the model Eq. (3.1), the point P_3^* is unstable.*

Proof. For the equilibrium P_3^*, the following is got

$$\mathbf{E} = \begin{pmatrix} \mu_2\alpha^2 M x_1^* & -(\mu_1 - \mu_2\alpha\beta M)x_1^* \\ (\mu_{10} + \mu_3\alpha\beta M)x_2^* & \mu_3\beta^2 M x_2^* \end{pmatrix}. \tag{3.15}$$

The corresponding characteristic equation is

$$\lambda^2 - D_1\lambda + D_2 = 0, \tag{3.16}$$

where $D_1 = (\mu_2\alpha^2 x_1^* + \mu_3\beta^2 x_2^*)M > 0$, $D_2 = (\mu_1\mu_{10} + (\mu_1\mu_3 - \mu_{10}\mu_2)\alpha\beta M x_1^* x_2^* > 0$.

The corresponding characteristic roots are

$$\lambda_1 = \frac{D_1 + \sqrt{D_1^2 - 4D_2}}{2}, \quad \lambda_2 = \frac{D_1 - \sqrt{D_1^2 - 4D_2}}{2} \tag{3.17}$$

Due to $\lambda_1 > 0$, $\lambda_2 > 0$, thus the equilibrium P_3^* is unstable.□

Remark 3.2. In Theorem 3.2, if $r < \mu_2\alpha d$, then the equilibrium $P_1^* = (0, 0, d)$ is stable, which indicates that the shrimp populations and the fish populations are about to extinct. Obviously, it is not the purpose of management.

In Theorem 3.3, if $\mu_2\alpha d < r$, the equilibrium $P_2^* = \left(\frac{\mu_2}{r} d - \frac{1}{\alpha}, 0, \frac{r}{\mu_2\alpha}\right)$ is stable, which indicates that the fish populations are about to extinct. It is not also the purpose of management.

In Theorem 3.4, if the equilibrium $P_3^* = \left(\frac{\mu_{20} + \mu_3\beta y^* 3_1}{\mu_{10}}, \frac{r - \mu_2\alpha y_1^*}{\mu_1}, y_1^*\right)$ is stable, which indicates that the shrimp populations and the fish populations are persistent; If the concentration of the cadmium is small enough, thus we think of the point P_3 as the ideal state of management.

In the following section, variable structure controller $\mathbf{u} = (u_1, u_2)^T$ is designed to make the system stable at equilibrium P_3^*.

4. Controller design

Consider the simplified model corresponding to the management model Eq. (2.9)

$$\begin{cases} \dot{x}_1 = rx_1 - \mu_1 x_1 x_2 - \mu_2\alpha x_1 y, \\ \dot{x}_2 = \mu_{10} x_1 x_2 - \mu_{20} x_2 - \mu_3\beta x_2 y - u_1, \\ 0 = y + \alpha x_1 y + \beta x_2 y - d - y u_2. \end{cases} \tag{4.1}$$

The corresponding matrix form is

$$\mathbf{E}\dot{\boldsymbol{\chi}} = \mathbf{A}(\boldsymbol{\chi}) + \mathbf{B}(\boldsymbol{\chi})\mathbf{u}, \tag{4.2}$$

where

$$\boldsymbol{\chi} = \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}, \quad \mathbf{A}(\boldsymbol{\chi}^T) = \begin{pmatrix} rx_1 - \mu_1 x_1 x_2 - \mu_2 \alpha x_1 y \\ \mu_{10} x_1 x_2 - \mu_{20} x_2 - \mu_3 \beta x_2 y \\ y + \alpha x_1 y + \beta x_2 y - d \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}(\boldsymbol{\chi}) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -y \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Before the controller is designed, the model Eq. (4.1) should be normalized.

4.1. Normalization of the model

Take

$$\mathbf{u} = -\mathbf{K}_2(\boldsymbol{\chi})\dot{\boldsymbol{\chi}} + \mathbf{u}^*, \tag{4.3}$$

where $\mathbf{u}^* = (u_1^*, u_2^*)^{\mathrm{T}}$ is a new control input of the model.

Choose

$$\mathbf{K}_2(\boldsymbol{\chi}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{y} \end{pmatrix}$$

such that

$$\det(\mathbf{E} + \mathbf{B}\mathbf{K}_2(\boldsymbol{\chi})) \neq 0.$$

thus, the normalized model corresponding to the model Eq. (4.1) is

$$\begin{cases} \dot{x}_1 = rx_1 - \mu_1 x_1 x_2 - \mu_2 \alpha x_1 y, \\ \dot{x}_2 = \mu_{10} x_1 x_2 - \mu_{20} x_2 - \mu_3 \beta x_2 y - u_1^*, \\ \dot{y} = y + \alpha x_1 y + \beta x_2 y - d - y u_2^*. \end{cases} \tag{4.4}$$

For the model Eq. (4.4), let $\bar{x}_1 = x_1 - x_1^*$, $\bar{x}_2 = x_2 - x_2^*$, $\bar{y} = y - y^*$, the following is got

$$\begin{cases} \dot{\bar{x}}_1 = r(\bar{x}_1 + x_1^*) - \mu_1(\bar{x}_1 + x_1^*)(\bar{x}_2 + x_2^*) - \mu_2 \alpha(\bar{x}_1 + x_1^*)(\bar{y} + y^*), \\ \dot{\bar{x}}_2 = \mu_{10}(\bar{x}_1 + x_1^*)(\bar{x}_2 + x_2^*) - \mu_{20}(\bar{x}_2 + x_2^*) - \mu_3 \beta(\bar{x}_2 + x_2^*)(\bar{y} + y^*) - u_1^*, \\ \dot{\bar{y}} = (\bar{y} + y^*) + \alpha(\bar{x}_1 + x_1^*)(\bar{y} + y^*) + \beta(\bar{x}_2 + x_2^*)(\bar{y} + y^*) - d - (\bar{y} + y^*) u_2^*. \end{cases} \tag{4.5}$$

The corresponding matrix form is

$$\dot{\bar{\boldsymbol{\chi}}} = \overline{\mathbf{A}}(\bar{\boldsymbol{\chi}}) + \overline{\mathbf{B}}(\bar{\boldsymbol{\chi}})\mathbf{u}^*, \tag{4.6}$$

where

$$\boldsymbol{\chi} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{y})^{\mathrm{T}}, \quad \widetilde{\mathbf{B}}(\widetilde{\boldsymbol{\chi}}) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -(\widetilde{y} + y^*) \end{pmatrix},$$

$$\begin{aligned} \widetilde{\mathbf{A}}(\widetilde{\boldsymbol{\chi}}) &= \begin{pmatrix} r(\widetilde{x}_1 + x_1^*) - \mu_1(\widetilde{x}_1 + x_1^*)(\widetilde{x}_2 + x_2^*) - \mu_2 \alpha(\widetilde{x}_1 + x_1^*)(\widetilde{y} + y^*) \\ \mu_{10}(\widetilde{x}_1 + x_1^*)(\widetilde{x}_2 + x_2^*) - \mu_{20}(\widetilde{x}_2 + x_2^*) - \mu_3 \beta(\widetilde{x}_2 + x_2^*)(\widetilde{y} + y^*) \\ (\widetilde{y} + y^*) + \alpha(\widetilde{x}_1 + x_1^*)(\widetilde{y} + y^*) + \beta(\widetilde{x}_2 + x_2^*)(\widetilde{y} + y^*) - d \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{A}_1(\widetilde{\boldsymbol{\chi}}) \\ \widetilde{A}_2(\widetilde{\boldsymbol{\chi}}) \\ \widetilde{A}_3(\widetilde{\boldsymbol{\chi}}) \end{pmatrix}. \end{aligned}$$

According to the literature [28], the following is got

$$\mathbf{R}(\overline{\boldsymbol{\chi}}) = \left(\overline{\mathbf{b}}_1, \overline{\mathbf{b}}_{1\overline{\mathbf{A}}}^{(1)}(\overline{\boldsymbol{\chi}}), \overline{\mathbf{b}}_2\right),$$

where $\overline{\mathbf{b}}_1, \overline{\mathbf{b}}_2$ are the first, the second column of the matrix $\overline{\mathbf{B}}(\overline{\boldsymbol{\chi}})$, respectively, and

$$\overline{\mathbf{b}}_{1\overline{\mathbf{A}}}^{(1)}(\overline{\boldsymbol{\chi}}) = \nabla\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}})\overline{\mathbf{b}}_1 - \nabla\overline{\mathbf{b}}_1\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}),$$

where $\nabla = \left(\frac{\partial}{\partial \overline{x}_1}, \frac{\partial}{\partial \overline{x}_2}, \frac{\partial}{\partial \overline{y}}\right)$.

That is

$$\mathbf{R}(\overline{\boldsymbol{\chi}}) = \begin{pmatrix} 0 & \mu_1(\overline{x}_1 + x_1^*) & 0 \\ -1 & -(\mu_{10}(\overline{x}_1 + x_1^*) - \mu_3\beta(\overline{y} + y^*) - \mu_{20}) & 0 \\ 0 & -\beta(\overline{y} + y^*) & -(\overline{y} + y^*) \end{pmatrix}.$$

where $\overline{x}_1 + x_1^* = x_1 \neq 0$.

Remark 4.1. $x_1 = 0$ means the shrimp populations are about to extinct, $x_1 \neq 0$ is the purpose of management.

Obviously, $\mathbf{R}(\overline{\boldsymbol{\chi}})$ is full rank, whose singular inverse matrix is

$$[\mathbf{R}(\overline{\boldsymbol{\chi}})]^{-1} = \begin{pmatrix} -\dfrac{\mu_{10}(\overline{x}_1 + x_1^*) - \mu_3\beta(\overline{y} + y^*) - \mu_{20}}{\mu_1(\overline{y} + y^*)} & -1 & 0 \\ \dfrac{1}{\mu_1(\overline{x}_1 + x_1^*)} & 0 & 0 \\ -\dfrac{\beta}{\mu_1(\overline{x}_1 + x_1^*)} & 0 & -\dfrac{1}{(\overline{y} + y^*)} \end{pmatrix}.$$

Let

$$[\mathbf{R}(\overline{\boldsymbol{\chi}})]_2^{-1} = \left(\frac{1}{\mu_1(\overline{x}_1 + x_1^*)}, 0, 0\right), \quad [\mathbf{R}(\overline{\boldsymbol{\chi}})]_3^{-1} = \left(-\frac{\beta}{\mu_1(\overline{x}_1 + x_1^*)}, 0, -\frac{1}{(\overline{y} + y^*)}\right)$$

the primitive functions of $[\mathbf{R}(\overline{\boldsymbol{\chi}})]_2^{-1}$ and $[\mathbf{R}(\overline{\boldsymbol{\chi}})]_3^{-1}$ are

$$\frac{1}{\mu_1}\ln(\overline{x}_1 + x_1^*) + c_1^* = \int [\mathbf{R}(\overline{\boldsymbol{\chi}})]_2^{-1} d\overline{\boldsymbol{\chi}},$$

$$-\frac{\beta}{\mu_1}\ln(\overline{x}_1 + x_1^*) - \ln(\overline{y} + y^*) + c_2^* = \int [\mathbf{R}(\overline{\boldsymbol{\chi}})]_3^{-1} d\overline{\boldsymbol{\chi}}.$$

Take

$$\begin{aligned} T_1(\overline{\boldsymbol{\chi}}) &= \int [\mathbf{R}(\overline{\boldsymbol{\chi}})]_2^{-1} d\overline{\boldsymbol{\chi}}, \\ T_2(\overline{\boldsymbol{\chi}}) &= [\mathbf{R}(\overline{\boldsymbol{\chi}})]_3^{-1}\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}), \\ T_3(\overline{\boldsymbol{\chi}}) &= \int [\mathbf{R}(\overline{\boldsymbol{\chi}})]_3^{-1} d\overline{\boldsymbol{\chi}}. \end{aligned} \tag{4.7}$$

Let

$$\mathbf{T}(\overline{\boldsymbol{\chi}}) = \begin{pmatrix} T_1(\overline{\boldsymbol{\chi}}) \\ T_2(\overline{\boldsymbol{\chi}}) \\ T_3(\overline{\boldsymbol{\chi}}) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\mu_1}\ln(\overline{x}_1 + x_1^*) + c_1^* \\ \dfrac{r}{\mu_1} - (\overline{x}_2 + x_{22}^*) - \dfrac{\mu_2\alpha}{\mu_1}(\overline{y} + y^*) \\ -\dfrac{\beta}{\mu_1}\ln(\overline{x}_1 + x_1^*) - \ln(\overline{y} + y^*) + c_2^* \end{pmatrix}. \tag{4.8}$$

According to the literature [28], choose the transformation matrix $\mathbf{T}(\overline{\boldsymbol{\chi}})$ and denote $\widehat{x}_1 = T_1(\overline{\boldsymbol{\chi}})$, $\widehat{x}_2 = T_2(\overline{\boldsymbol{\chi}})$, $\widehat{y} = T_3(\overline{\boldsymbol{\chi}}), \widehat{\boldsymbol{\chi}} = (\widehat{x}_1, \widehat{x}_2, \widehat{y})^{\mathrm{T}}$, the following theorem is obtained.

Theorem 4.1. *For the model Eq. (4.6), the state transformation* $\boldsymbol{\chi} \to \widehat{\boldsymbol{\chi}} : \widehat{\boldsymbol{\chi}} = \mathbf{T}(\boldsymbol{\chi})$:

$$\widehat{\boldsymbol{\chi}} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} T_1(\overline{\boldsymbol{\chi}}) \\ T_2(\overline{\boldsymbol{\chi}}) \\ T_3(\overline{\boldsymbol{\chi}}) \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu_1}\ln\left(\overline{x}_1 + x_1^*\right) + c_1^* \\ \frac{r}{\mu_1} - \left(\overline{x}_2 + x_{22}^*\right) - \frac{\mu_2\alpha}{\mu_1}\left(\overline{y} + y^*\right) \\ -\frac{\beta}{\mu_1}\ln\left(\overline{x}_1 + x_1^*\right) - \ln\left(\overline{y} + y^*\right) + c_2^* \end{pmatrix} \tag{4.9}$$

makes the model into

$$\begin{pmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \\ \dot{\widehat{y}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_1(\overline{\boldsymbol{\chi}}) \\ \eta_2(\overline{\boldsymbol{\chi}}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & \frac{\mu_2\alpha}{\mu_1}\left(\overline{y} + y^*\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}, \tag{4.10}$$

where

$$\eta_1(\overline{\boldsymbol{\chi}}) = -\overline{A}_2(\overline{\boldsymbol{\chi}}) - \frac{\mu_2\alpha}{\mu_1}\overline{A}_3(\overline{\boldsymbol{\chi}}), \quad \eta_2(\overline{\boldsymbol{\chi}}) = -\frac{\beta}{\mu_1\left(\overline{y} + y^*\right)}\overline{A}_1(\overline{\boldsymbol{\chi}}) - \frac{1}{\left(\overline{y} + y^*\right)}\overline{A}_3(\overline{\boldsymbol{\chi}}).$$

Proof. Differentiate the formula Eq. (4.9), the following is given

$$\begin{pmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \\ \dot{\widehat{y}} \end{pmatrix} = \begin{pmatrix} \dot{T}_1(\overline{\boldsymbol{\chi}}) \\ \dot{T}_2(\overline{\boldsymbol{\chi}}) \\ \dot{T}_3(\overline{\boldsymbol{\chi}}) \end{pmatrix} = \begin{pmatrix} \nabla T_1(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}) \\ \nabla T_2(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}) \\ \nabla T_3(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}) \end{pmatrix} + \begin{pmatrix} \nabla T_1\left(\overline{\boldsymbol{\chi}}^T\right)\overline{\mathbf{B}}\left(\overline{\boldsymbol{\chi}}^T\right) \\ \nabla T_2\left(\overline{\boldsymbol{\chi}}^T\right)\overline{\mathbf{B}}\left(\overline{\boldsymbol{\chi}}^T\right) \\ \nabla T_3\left(\overline{\boldsymbol{\chi}}^T\right)\overline{\mathbf{B}}\left(\overline{\boldsymbol{\chi}}^T\right) \end{pmatrix}\mathbf{u}^*. \tag{4.11}$$

where

$$\nabla T_1(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}\left(\overline{\boldsymbol{\chi}}^T\right) = \frac{r}{\mu_1} - \left(\overline{x}_2 + x_{22}^*\right) - \frac{\mu_2\alpha}{\mu_1}\left(\overline{y} + y^*\right), \quad \nabla T_2(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}) = -\overline{A}_2(\overline{\boldsymbol{\chi}}) - \frac{\mu_2\alpha}{\mu_1}\overline{A}_3(\overline{\boldsymbol{\chi}}),$$

$$\nabla T_3(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}) = -\frac{\beta}{\mu_1\left(\overline{y} + y^*\right)}\overline{A}_1(\overline{\boldsymbol{\chi}}) - \frac{1}{\left(\overline{y} + y^*\right)}\overline{A}_3(\overline{\boldsymbol{\chi}}),$$

$$\nabla T_1(\overline{\boldsymbol{\chi}})\overline{\mathbf{B}}(\overline{\boldsymbol{\chi}}) = (0,0), \quad \nabla T_2(\overline{\boldsymbol{\chi}})\overline{\mathbf{B}}(\overline{\boldsymbol{\chi}}) = \left(1, \frac{\mu_2\alpha}{\mu_1}\left(\overline{y} + y^*\right)\right), \quad \nabla T_3(\overline{\boldsymbol{\chi}})\overline{\mathbf{B}}(\overline{\boldsymbol{\chi}}) = (0,1).$$

Obviously, $\nabla T_1(\overline{\boldsymbol{\chi}})\overline{\mathbf{A}}(\overline{\boldsymbol{\chi}}) = T_2(\overline{\boldsymbol{\chi}}) = \widehat{x}_2$.
Let

$$\eta_1(\overline{\boldsymbol{\chi}}) = -\overline{A}_2(\overline{\boldsymbol{\chi}}) - \frac{\mu_2\alpha}{\mu_1}\overline{A}_3(\overline{\boldsymbol{\chi}}),$$

$$\eta_2(\overline{\boldsymbol{\chi}}) = -\frac{\beta}{\mu_1\left(\overline{y} + y^*\right)}\overline{A}_1(\overline{\boldsymbol{\chi}}) - \frac{1}{\left(\overline{y} + y^*\right)}\overline{A}_3(\overline{\boldsymbol{\chi}}).$$

Thus, the controllability canonical form of the model Eq. (2.3) is

$$\begin{pmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \\ \dot{\widehat{y}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_1(\overline{\boldsymbol{\chi}}) \\ \eta_2(\overline{\boldsymbol{\chi}}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & \frac{\mu_2\alpha}{\mu_1}\left(\overline{y} + y^*\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}.$$

□ Based on the above, the controller is designed in the following section.

4.2. Controller design

According to the theory of variable structure control, controller is designed not only to make the shrimp populations and the fish populations persist, but also to reduce the concentration of cadmium in the recommended limits for human consumption as soon as possible. The specific steps are as follows:

- **Step 1** Choice of switching function
  (1) The model Eq. (4.10) is decomposed into the following models

$$\begin{pmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_1(\overline{\chi}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & \frac{\mu_2\alpha}{\mu_1}(\overline{y}+y^*) \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}. \tag{4.12}$$

$$\dot{\widehat{y}} = \widehat{y} + \eta_2(\overline{\chi}) + u_2^*. \tag{4.13}$$

(2) Take the switching function

$$\mathbf{s}(\overline{\chi}) = (s_1(\overline{\chi}), \quad s_2(\overline{\chi}))^{\mathrm{T}} = (c\widehat{x}_1 + \widehat{x}_2, \quad \widehat{y})^{\mathrm{T}} \tag{4.14}$$

where c is a constant.

- **Step 2** Design of controller
  Let $s_1(\overline{\chi}) = 0, \ s_2(\overline{\chi}) = 0$ and combine $s_1(\overline{\chi}) = 0$ with the model Eq. (4.10), the following is got

$$\begin{cases} \widehat{x}_1 = e^{-ct}, \\ \widehat{x}_2 = -ce^{-ct}, \\ \widehat{y} = 0. \end{cases} \tag{4.15}$$

If $c > 0$, then $\lim\limits_{t\to\infty}(\widehat{x}_1,\widehat{x}_2,\widehat{y}) = (0,0,0)$, that is, the sliding mode $\mathbf{s}(\overline{\chi}) = 0$ of the model Eq. (4.10) is asymptotically stable at the point (0,0,0).

Differentiate $s_1(\overline{\chi})$ along the model Eq. (4.11), the following is got

$$\dot{s}_1(\overline{\chi}) = c\dot{\widehat{x}}_1 + \dot{\widehat{x}}_2 = c\widehat{x}_2 + \eta_1(\overline{\chi}) + u_1^* + \frac{\mu_2\alpha}{\mu_1}(\overline{y}+y^*)u_2^*, \tag{4.16}$$

Differentiate $s_2(\overline{\chi})$ along the model Eq. (4.12), the following is got

$$\dot{s}_2(\overline{\chi}) = \dot{\widehat{y}} = \eta_2(\overline{\chi}) + u_2^*. \tag{4.17}$$

Take the exponential approach laws

$$\dot{s}_1(\overline{\chi}) = -\varepsilon_1 \mathrm{sgn} s_1(\overline{\chi}) - k_1 s_1(\overline{\chi}), \tag{4.18}$$

$$\dot{s}_2(\overline{\chi}) = -\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}), \tag{4.19}$$

where $\varepsilon_1, \ \varepsilon_2, \ k_1, \ k_2$ are positive.

It is easy to prove that $s_1(\overline{\chi})\dot{s}_1(\overline{\chi}) < 0$ and $s_2(\overline{\chi})\dot{s}_2(\overline{\chi}) < 0$, thus $\mathbf{s}(\overline{\chi}) = 0$ is global stable.

According to the literature [28], combine the formula Eq. (4.15) with the formula Eq. (4.17); And combine the formula Eq. (4.16) with the formula Eq. (4.18), respectively, the controller $\mathbf{u}^* = (u_1^*, u_2^*)^{\mathrm{T}}$ of the model Eq. (4.10) is taken as

$$\begin{aligned} u_1^* \quad = \quad & -\varepsilon_1 \mathrm{sgn} s_1(\overline{\chi}) - k_1 s_1(\overline{\chi}) - c\widehat{x}_2 - \eta_1(\overline{\chi}) \\ & - \frac{\mu_2\alpha}{\mu_1}(\overline{y}+y^*)(-\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi})) \end{aligned} \tag{4.20}$$

$$u_2^* = -\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi}). \tag{4.21}$$

Remark 4.2. In Step 2, because $\mathbf{s}(\overline{\chi}) = 0$ is global stable, take the sliding mode as the switching surface. It is worth noting that they may be different.

According to the steps above, the main theorems of the paper are obtained.

Theorem 4.2. *For the model Eq. (4.10), the control*

$$\mathbf{u}^* = \begin{pmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^* \end{pmatrix}$$

$$= \begin{pmatrix} -\varepsilon_1 \mathrm{sgn} s_1(\overline{\chi}) - k_1 s_1(\overline{\chi}) - c\widehat{x}_2 - \eta_1(\overline{\chi} - \frac{\mu_2\alpha}{\mu_1}(\overline{y}+y^*)(-\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi})) \\ -\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi}). \end{pmatrix}$$

makes the trajectories of the model reach the switching surface $\mathbf{s}(\overline{\chi}) = \mathbf{0}$ *in finite time*

$$t^* = \max\left(\frac{1}{k_1}\ln\left(\frac{\varepsilon_1 + k_1|s_{10}|}{\varepsilon_1}\right), \frac{1}{k_2}\ln\left(\frac{\varepsilon_2 + k_2|s_{20}|}{\varepsilon_2}\right)\right).$$

where $\varepsilon_1, \varepsilon_2, k_1, k_2$ are parameters.

Proof. For the model Eq. (4.11), the following two cases are discussed

case 1 $s_1(\overline{\chi}) > 0$

If $s_1(\overline{\chi}) > 0$, then

$$\dot{s}_1(\overline{\chi}) = -\varepsilon_1 - k_1 s_1(\overline{\chi}).$$

The corresponding solution is

$$s_1(\overline{\chi}) = \left(s_{10} + \frac{\varepsilon_1}{k_1}\right)e^{-k_1 t_1} - \frac{\varepsilon_1}{k_1},$$

where s_{10} is the initial value of $s_1(\overline{\chi})$.

Let $s_1(\overline{\chi}) = 0$, the followings are got

$$t_1 = \overline{t}_1^* = \frac{1}{k_1}\ln\left(\frac{\varepsilon_1 + k_1 s_{10}}{\varepsilon_1}\right), \quad \dot{s}_1(\overline{\chi}) = -\varepsilon_1$$

case 2 $s_1(\overline{\chi}) < 0$

If $s_1(\overline{\chi}) < 0$, then

$$\dot{s}_1(\overline{\chi}) = \varepsilon_1 - k_1 s_1(\overline{\chi}).$$

The corresponding solution is

$$s_1(\overline{\chi}) = \left(s_{10} - \frac{\varepsilon_1}{k_1}\right)e^{-k_1 t_1} + \frac{\varepsilon_1}{k_1}.$$

Let $s_1(\overline{\chi}) = 0$, the followings are got

$$t_1 = \overline{t}_1^{**} = \frac{1}{k_1}\ln\left(\frac{\varepsilon_1 + k_1 s_{10}}{\varepsilon_1}\right), \quad \dot{s}_1(\overline{\chi}) = -\varepsilon_1$$

In short, the trajectories of the model Eq. (4.10) may reach the surface $s_1(\overline{\chi}) = 0$ in finite time $t_1^* = \max\{\widetilde{t}_1^*, \widetilde{t}_1^{**}\} = \frac{1}{k_1}\ln\left(\frac{\varepsilon_1 + k_1|s_{10}|}{\varepsilon_1}\right)$, where s_{10} is the initial value of $s_1(\overline{\chi})$.

Similarly, the trajectories of the model Eq. (4.10) may reach the surface $s_2(\overline{\chi}) = 0$ in finite time $t_2^* = \frac{1}{k_2}\ln\left(\frac{\varepsilon_2 + k_2|s_{20}|}{\varepsilon_2}\right)$, where s_{20} is the initial value of $s_2(\overline{\chi})$.

Let $t^* = \max\{t_1^*, t_2^*\} = \max\left(\frac{1}{k_1}\ln\left(\frac{\varepsilon_1 + k_1|s_{10}|}{\varepsilon_1}\right), \frac{1}{k_2}\ln\left(\frac{\varepsilon_2 + k_2|s_{20}|}{\varepsilon_2}\right)\right)$, thus the trajectories of the model Eq. (4.10) may reach the switching surface $\mathbf{s}(\overline{\chi}) = \mathbf{0}$ in finite time t^*.□

Remark 4.3. If ε_1, ε_2 are reduced, then the trajectories of the model Eq. (4.10) enters into the switching surface $\mathbf{s}(\overline{\chi}) = \mathbf{0}$ as soon as possible; If k_1, k_2 are increased, the corresponding vibration is weaken.

Further, the following theorem is obtained.

Theorem 4.3. *For the model Eq. (4.10), if the conditions in Theorem 4.2 are satisfied, then the model is asymptotically stable at the point (0,0,0).*

According to the control Eq. (4.3), the variable structure controller $\mathbf{u} = (u_1, u_2)$ of the model Eq. (4.1) is obtained

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\varepsilon_1 \mathrm{sgn} s_1(\overline{\chi}) - k_1 s_1(\overline{\chi}) - c\widehat{x}_2 - \eta_1(\overline{\chi}) - \dfrac{\mu_2\alpha}{\mu_1}(\overline{y} + y^*)(-\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi})) \\ -\dfrac{1}{y}\dot{y} - \varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi}). \end{pmatrix}$$

Thus, the following is got:

Theorem 4.4. *For the model Eq. (4.1), the variable structure control*

Table 5.1
The values of coefficients.

r	μ_1	μ_{10}	μ_{20}	d	α	β	μ_2	μ_3	c	k_1	k_2	ε_1	ε_2
0.66	0.3	0.25	0.1	0.5	150	300	0.0001	0.0015	1	0.3	0.1	0.2	0.1

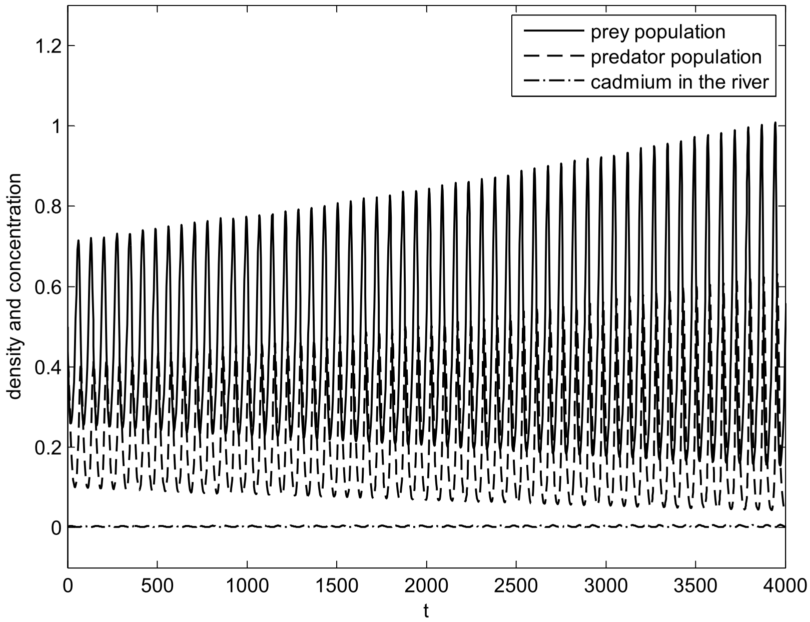

Fig. 4.1. Time response of the model Eq. (3.1) at the equilibrium P_3^*.

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\varepsilon_1 \mathrm{sgn} s_1(\overline{\chi}) - k_1 s_1(\overline{\chi}) - c\widehat{x}_2 - \eta_1(\overline{\chi}) - \dfrac{\mu_2 \alpha}{\mu_1}(\overline{y} + y^*)(-\varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi})) \\ -\dfrac{1}{y}\dot{y} - \varepsilon_2 \mathrm{sgn} s_2(\overline{\chi}) - k_2 s_2(\overline{\chi}) - \eta_2(\overline{\chi}). \end{pmatrix}$$

makes the trajectory of the model reach the switching surface $\mathbf{s}(\overline{\chi}) = \mathbf{0}$ *in finite time*

$$t = \max\left(\frac{1}{k_1}\ln\left(\frac{\varepsilon_1 + k_1|s_{10}|}{\varepsilon_1}\right), \frac{1}{k_2}\ln\left(\frac{\varepsilon_2 + k_2|s_{20}|}{\varepsilon_2}\right)\right),$$

and the model with small y^* *stable at equilibrium* P_3^*. *Where* $\varepsilon_1 > 0, \varepsilon_2 > 0, k_1 > 0, k_2 > 0, c > 0,$ In the next section, the further discussions are given.

5. Discussions

Take coefficients of the model Eq. (3.1) and the model Eq. (4.1) as follows (See Table 5.1).

By MATLAB Software, the time responses of the model Eq. (3.1) and the model Eq. (4.1) at the equilibrium P_3^* are got.

Theorem 3.4 shows that the existence of the cadmium in the environment leads to the extinction of fish populations as time goes by (See Fig. 4.1). On the other hand, it shows that the cadmium in the environment has more effect on the fish populations than the shrimp populations if the cadmium increases abruptly. In order to avoid the phenomenon, the controller is designed by regulating the cadmium in the body of fish populations and in the environment (See Theorem 4.2).

The result (In Theorem 4.2) shows that the model Eq. (4.1) with control is stable at the equilibrium P_3^*, which means that the shrimp populations and the fish populations stabilize in the ideal state as quickly as possible and the concentration of the cadmium in the environment reduces to the environmental quality standard as soon as possible (See Fig. 4.2).

In short, the study in the paper shows the following two highlights: One is that the fast-slow process such as a one-off discharge and bioconcentration on tissues and organs of toxic substances is described by a singular predator-prey model; the other is that the variable

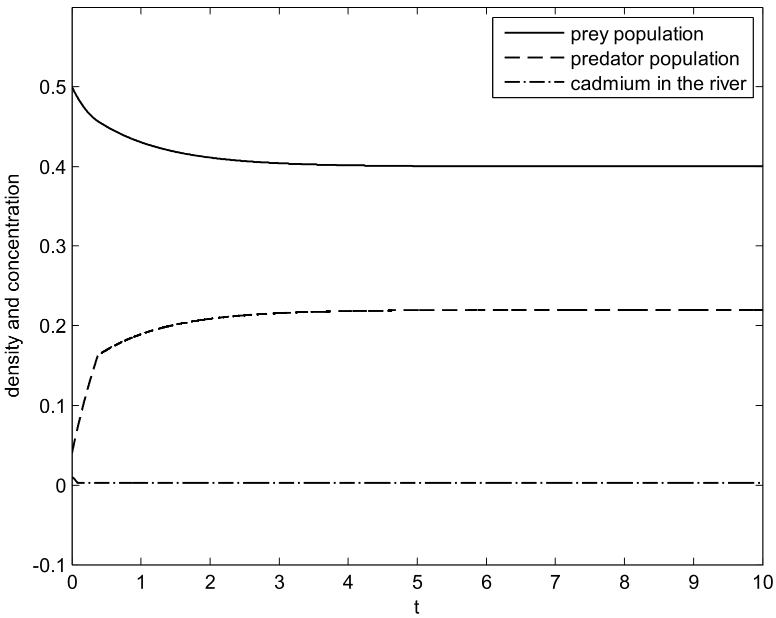

Fig. 4.2. Time response of the model Eq. (4.1) at the equilibrium P_3^*.

structure control is used to make the system in the ideal state as quickly as possible and cut the cost of control. However, whether a one-off discharge of toxic substances leads to the catastrophe of the system and whether the variable structure control is used to control the catastrophe, which is our work in the future.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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