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Stationary Distribution of a Stochastic Hybrid Model with Fear: a Markov Semigroup Theory

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Abstract In this paper, we investigate the stationary distribution of a novel stochastic hybrid predator-prey model with fear effect and Beddington-DeAngelis functional response. Based on Markov semigroup theory, the existence of asymptotically stable stationary distribution of stochastic hybrid system is established, which can converge in L^1 to an invariant density under appropriate conditions. Moreover, we derive sufficient conditions for extinction and persistence of prey and predator populations in the stochastic hybrid model.

Keywords stochastic hybrid model; fear effect; Markov semigroup; stationary distribution; extinction and persistence

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1 Introduction

Recently, the stochastic population models affected by fear effect have become a hot issue being researched. Fear is the inherent psychological reaction of biological organisms, which can help organisms to improve their alertness and response capability, thereby avoiding danger, like the fear of prey population being hunted by predator population, but it also affects foraging behavior[2], reproductive ability and strategy[7], habitat utilization[8], and the birth rate and survival rate of offspring[28]. In 2021, Qi and Meng[19] purposed a stochastic predator-prey system with fear effect, and the threshold for the existence of a unique ergodic stationary distribution is established. Liu et al.[16] also considered the effect of fear on the predator-prey model and obtained the existence of ergodic stationary distribution. They both studied the existence and stability of stationary distribution, which is a key issue worthy of attention for the study of stochastic differential dynamic models. Similarly, in [16, 20, 21], they also employed the traditional method of stochastic analysis and Khasminskiĭ ergodicity theory[11, 17] to analyze the existence of ergodic stationary distribution. But this method has certain limitations, that is, degradation diffusion cannot be handled well. Therefore, to deal with these limitations, Rudnicki et al.[25] utilized the Markov semigroup theory to discuss the stationary distribution of some stochastic predator-prey models[14, 23, 24] and stochastic epidemic models[6, 12, 15]. For example, Lin and Jiang[14] considered a stochastic predator-prey model with modified Leslie-Gower and Holling-type II schemes, they gave a sufficient condition for the existence of a stationary distribution.

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However, in [4, 6, 9, 10, 14, 16, 19–21, 23, 24, 29], they only considered the impact of white noise on the predator-prey models or stochastic epidemic models. As far as we know, the predator-prey models or epidemic models can also be affected by the color noise, which is described by continuous time Markov chain in mathematics[12, 15, 27]. For example, in [15], Lin et al. considered a regime-switching SIRS epidemic model with degenerate diffusion, they found that the densities of the distributions of the solution can converge in L^1 to an invariant density under certain conditions. Lan et al.[12] considered a stochastic SIRS epidemic model with non-monotone incidence rate under regime-switching, they also found that the densities of the distributions of the solution can converge in L^1 to an invariant density by using the Markov semigroups theory. They both only considered the use of Markov semigroup theory to analyze the characteristics of epidemic models with regime-switching but neglected to study the dynamic properties of stochastic population models with regime-switching. Hence, we propose a novel stochastic hybrid predator-prey model with fear effect as follows

$$\begin{cases} dx(t) = x\Big[r(\xi(t))\big(\eta(\xi(t)) + \dfrac{\alpha(\xi(t))(1-\eta(\xi(t)))}{\alpha(\xi(t))+y}\big) - \mu(\xi(t)) - \gamma(\xi(t))x \\ \qquad\qquad - \dfrac{\beta(\xi(t))y}{1+a(\xi(t))x+b(\xi(t))y}\Big]\,dt + \sigma(\xi(t))x\,dB(t), \\ dy(t) = y\Big[\dfrac{\theta(\xi(t))\beta(\xi(t))x}{1+a(\xi(t))x+b(\xi(t))y} - d(\xi(t)) - \delta(\xi(t))y\Big]\,dt + \sigma(\xi(t))y\,dB(t), \end{cases} \tag{1.1}$$

where x and y denote the density of prey and predator population, respectively. r denotes the birth rate of prey, η represents the cost of minimum fear and $\eta \in [0,1]$, α stands for the level of fear with prey population, γ and δ represent the intra-species competition of prey and predatory, respectively. μ, d, β, θ represent the prey death rate, the predator death rate, the predation rate, the conversion rate, respectively. $\frac{x}{1+ax+by}$ stands for the Beddington-DeAngelis functional response. All parameter values of model (1.1) are positive constants for any $k \in \mathbb{M} \triangleq \{1,2,\cdots,N\}$. $B(t)$ is a standard Brownian motion denoted on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $\sigma > 0$ stands for the intensities of the white noise. $\xi(t)$ is a continuous time Markov chain with values in finite state space $\mathbb{M}$.

For any vector $g = (g(1), g(2), \cdots, g(N))$, we let $\hat{g} = \min_{k\in\mathbb{M}}\{g(k)\}$ and $\check{g} = \max_{k\in\mathbb{M}}\{g(k)\}$. Suppose the generator $\Gamma = (\gamma_{kl})_{N\times N}$ of the Markov chain is given by

$$\mathbb{P}(\xi(t+\Delta) = l \mid \xi(t) = k) = \begin{cases} \gamma_{kl}\Delta + o(\Delta), & \text{if} \quad k \neq l, \\ 1 + \gamma_{kl}\Delta + o(\Delta), & \text{if} \quad k = l, \end{cases}$$

where $\Delta > 0$, $\gamma_{kl} \geq 0$ is the transition rate from k to l if $k \neq l$ while $\sum_{l=1}^{N}\gamma_{kl} = 0$. In this paper, we assume Markov chain $\xi(t)$ is irreducible and independent of Brownian motion and assume $\gamma_{kl} \geq 0$, for any $k \neq l$. Assume further that Markov chain $\xi(t)$ is irreducible and has a unique stationary distribution $\pi = \{\pi_1, \pi_2, \cdots, \pi_N\}$ which can be determined by equation $\pi\Gamma = 0$ subject to $\sum_{k=1}^{N}\pi_k = 1$, and $\pi_k > 0$, $\forall\, k \in \mathbb{M}$.

Let $u = \ln x$, $v = \ln y$. Then, by Itô's formula, the random variables u, v satisfy

$$\begin{cases} du(t) = \Big[h_1(k) + \dfrac{r(k)\alpha(k)(1-\eta(k))}{\alpha(k)+e^v} - \gamma(k)e^u - \dfrac{\beta(k)e^v}{1+a(k)e^u+b(k)e^v}\Big]\,dt \\ \qquad\qquad +\sigma(k)\,dB(t), \\ dv(t) = \Big[\dfrac{\theta(k)\beta(k)e^u}{1+a(k)e^u+b(k)e^v} - h_2(k) - \delta(k)e^v\Big]\,dt + \sigma(k)\,dB(t), \end{cases} \tag{1.2}$$

where $h_1(k) = r(k)\eta(k) - \mu(k) - \frac{\sigma^2(k)}{2}$, $h_2(k) = d(k) + \frac{\sigma^2(k)}{2}$.

Obviously, the model (1.1) is equivalent to the model (1.2), then in this paper, some definitions and results of Markov semigroup, and some useful Lemma are listed in Section 2. We investigate the existence and stability of stationary distribution of model (1.2) by using Markov semigroup theory, which is proved in Section 3. Section 4 gives the extinction and persistence of the stochastic hybrid model (1.1).

2 Preliminary

In this section, we provide some auxiliary definitions and results about Markov semigroup (see [11, 16, 19]).

Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0, 1 \leq i \leq n\}$. Let $\mathbb{X} = \mathbb{R}^2_+$, $\Sigma = \mathfrak{B}$ be the σ-algebra of Borel subset of $\mathbb{X}$ and m be the Lebesgue measure on $(\mathbb{X}, \Sigma)$. Let the triple $(\mathbb{X}, \Sigma, m)$ be a σ-finite measure space. Denote $\mathbb{D}$ be the subset of the space $L^1 = L^1(\mathbb{X}, \Sigma, m)$ which contains all densities, i.e.

$$\mathbb{D} = \{f \in L^1 : f \geq 0, ||f|| = 1\}.$$

A linear mapping $\mathcal{P} : L^1 \to L^1$ is called a Markov operator if $\mathcal{P}(\mathbb{D}) \subset \mathbb{D}$.

The Markov operator $\mathcal{P}$ is called an integral or kernel operator if there exists a measurable function $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ such that

$$\mathcal{P}f(x) = \int_{\mathbb{X}} \mathcal{K}(x, y) f(y) m(\,dy)$$

for every density f. One can check that from the condition $\mathcal{P}(\mathbb{D}) \subset \mathbb{D}$ it follows that

$$\int_{\mathbb{X}} \mathcal{K}(x, y) m(dx) = 1$$

for all $y \in \mathbb{X}$.

A family $\{\mathcal{P}(t)\}_{t\geq 0}$ of Markov operators which satisfies conditions:

(a) $\mathcal{P}(0) = \mathbf{I}\mathrm{d}$, ($\mathbf{I}$d denotes identity matrix);

(b) $\mathcal{P}(t + s) = \mathcal{P}(t)\mathcal{P}(s)$ for s, $t \geq 0$;

(c) The function $t \to \mathcal{P}(t)f$ is continuous with respect to the L^1 norm for each $f \in L^1$,

is called a Markov semigroup. A Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is called partially integral, if there exists $t_0 > 0$, and a measurable function $\mathcal{K} : (0, \infty) \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$, called a kernel, such that

$$\iint \mathcal{K}(x, y) m(dx) m(\,dy) > 0, \tag{2.1}$$

and

$$\mathcal{P}(t_0) f(x) \geq \int_{\mathbb{X}} \mathcal{K}(x, y) f(y) m(\,dy), \qquad \text{for every } f \in \mathbb{D}. \tag{2.2}$$

A density f_* is called invariant if $\mathcal{P}(t) f_* = f_*$ for each $t > 0$. The Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t\to\infty} ||\mathcal{P}(t) f - f_*|| = 0, \qquad \text{for } f \in \mathbb{D}.$$

A Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if for every $f \in \mathbb{D}$

$$\lim_{t\to\infty} \int_A \mathcal{P}(t)f(x)m(dx) = 0.$$

The following lemma summarizes some result concerning asymptotic stability and sweeping.

Lemma 2.1[16, 19]**.** *Let $\mathbb{X}$ be a metric space and Σ be the σ-algebra of Borel sets. Let $\{\mathcal{P}(t)\}_{t\geq 0}$ be an integral Markov semigroup with a continuous kernel $\mathcal{K}(t;x;y)$ for $t>0$, which satisfies (2.1)–(2.2) for all $y \in \mathbb{X}$. We assume that for every $f \in \mathbb{D}$ we have*

$$\int_0^\infty \mathcal{P}(t)f\,dt > 0, \qquad \text{a.e.}$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

The property that a Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable or sweeping from a sufficiently large family of sets is called the Foguel alternative.

3 Asymptotic Stability of Stationary Distribution

In this section, we utilize the Markov semigroup theory to investigate the existence and stability of stationary distribution of the model (1.2).

Denote $\mathbb{E} = \mathbb{R}^2_+ \setminus U$,

$$U = \Big\{(u,v) \in \mathbb{R}^2 : \gamma(k)e^u + \frac{r(k)\alpha(k)(1-\eta(k))e^v}{(\alpha(k)+e^v)^2} + \frac{(1+\theta(k))\beta(k)e^v}{(1+a(k)e^u+b(k)e^v)^2} - \delta(k)e^v = 0\Big\}.$$

Theorem 3.1. *Let $(u(t),v(t),\xi(t))$ be a solution of model (1.2) with any given initial value $(u(0),v(0),\xi(0)) \in \mathbb{E}\times\mathbb{M}$. Then for every $t>0$, the distribution of $(u(t),v(t),\xi(t))$ has a density $\pi(t,u,v,k)$. And if $\hat{\gamma} > \frac{\check{a}\check{\beta}y^*}{1+\hat{a}x^*+\hat{b}y^*} + \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}$, $\hat{\delta} > \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}$ and for $k \in \mathbb{M}$, $\theta(k)h_1(k)+h_2(k)>0$, and*

$$\sum_{k=1}^N \pi_k\kappa_k < \min\Big\{\Big(\hat{\gamma} - \frac{\check{a}\check{\beta}y^*}{1+\hat{a}x^*+\hat{b}y^*} - \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)x^{*2}, \Big(\hat{\delta} - \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)y^{*2}\Big\},$$

where $\sum_{k=1}^N \pi_k\kappa_k = \frac{1}{2}\sigma^2(k)x^* + \frac{\sigma^2(k)(1+a(k)x^*)y^*}{2\theta(k)(1+b(k)y^*)}$, (x^*,y^*) is the positive equilibrium of model (1.1) without noise interference, then there exists a unique density $\pi_*(t,u,v,k)$ such that

$$\lim_{t\to\infty} \sum_{k=1}^N \iint_{\mathbb{R}^2_+} |\pi(t,u,v,k) - \pi_*(t,u,v,k)|\,du\,dv = 0,$$

that is, the semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable.

Proof. The proof strategy for Theorem 3.1 can be shown in four steps:

Step 1. Applying the Hörmander Theorem[5], we show that the semigroup is a partially integral Markov semigroup, which corresponds to the following Lemma 3.2.

Lemma 3.2. *The semigroup $\{\mathcal{T}_k(t)\}_{t\geq 0}$ is an integral Markov semigroup.*

Proof. Let $\mathbf{a}(x)$ and $\mathbf{b}(x)$ are vector fields on $\mathbb{R}^d$, then the Lie bracket $[\mathbf{a}, \mathbf{b}]$ is a vector field given by

$$[\mathbf{a}, \mathbf{b}]_j(x) = \sum_{k=1}^{d} \Big(\mathbf{a}_k \frac{\partial \mathbf{b}_j}{\partial x_k}(x) - \mathbf{b}_k \frac{\partial \mathbf{a}_j}{\partial x_k}(x) \Big), \qquad j = 1, 2, \cdots, d.$$

Let

$$\mathbf{a}_1(u,v) = \begin{pmatrix} h_1(k) + \dfrac{r(k)\alpha(k)(1-\eta(k))}{\alpha(k)+e^v} - \gamma(k)e^u - \dfrac{\beta(k)e^v}{1+a(k)e^u+b(k)e^v} \\ \dfrac{\theta(k)\beta(k)e^u}{1+a(k)e^u+b(k)e^v} - h_2(k) \end{pmatrix},$$

$$\mathbf{a}_2(u,v) = \begin{pmatrix} \sigma(k) \\ \sigma(k) \end{pmatrix}.$$

Then

$$[\mathbf{a}_1, \mathbf{a}_2] = \begin{pmatrix} \gamma(k)\sigma(k)e^u + \dfrac{r(k)\alpha(k)\sigma(k)(1-\eta(k))e^v}{(\alpha(k)+e^v)^2} + \dfrac{\beta(k)\sigma(k)e^v}{(1+a(k)e^u+b(k)e^v)^2} \\ -\dfrac{\theta(k)\beta(k)\sigma(k)e^v}{(1+a(k)e^u+b(k)e^v)^2} + \sigma(k)\delta(k)e^v \end{pmatrix}.$$

Consequently,

$$\begin{aligned} &\big|\mathbf{a}_2 \; [\mathbf{a}_1, \mathbf{a}_2]\big| \\ =&\Big(-\gamma(k)e^u - \frac{r(k)\alpha(k)(1-\eta(k))e^v}{(\alpha(k)+e^v)^2} - \frac{(1+\theta(k))\beta(k)e^v}{(1+a(k)e^u+b(k)e^v)^2} + \delta(k)e^v \Big)\sigma^2(k) \neq 0, \end{aligned}$$

which implies that $\mathbf{a}_2(u,v)$, $[\mathbf{a}_1, \mathbf{a}_2](u,v)$ are linearly independent. Hence, for every $(u,v) \in \mathbb{E}$, vectors $\mathbf{a}_2(u,v)$, $[\mathbf{a}_1, \mathbf{a}_2](u,v)$ span the space $\mathbb{E}$. In view of Hörmander Theorem[5], there exists a smooth density $\mathcal{K}(t,u,v;u_0,v_0) \in C^\infty((0,\infty)\times\mathbb{E}\times\mathbb{E})$ such that for every $f \in L^1(\mathbb{E}, \mathcal{B}(\mathbb{E}), m)$ satisfying $f \geq 0$ and $||f|| = 1$,

$$\mathcal{T}_k(t)f(\xi,\zeta) = \iint_{\mathbb{E}} \mathcal{K}(t,u,v;\xi,\zeta)f(\xi,\zeta)\,d\xi\,d\zeta.$$

Then, the semigroup $\{\mathcal{T}_k(t)\}_{t\geq 0}$ is an integral Markov semigroup. □

Step 2. We prove that the density of the transition function is positive on $\mathbb{R}^2_+$ via using support theorems[1, 3, 26], which corresponds to Lemma 3.3.

Lemma 3.3. *If $\theta(k)h_1(k) + h_2(k) > 0$, $k \in \mathbb{M}$, then for any $f \in \mathbb{U}$,*

$$\int_0^\infty \mathcal{T}_k(t)f dt > 0, \quad \text{a.e. on } \mathbb{E},$$

where $\mathbb{U} = \{f \in L^1(\mathbb{E}, \mathcal{B}(\mathbb{E}), m) : f \geq 0, ||f|| = 1\}$.

Proof. The Itô's SDEs in model (1.2) can be rewritten in the Stratonovitch's form

$$\begin{cases} du_t = & f_1(u_t, v_t, \xi(t))\,dt + \sigma(k) \circ dB_t, \\ dv_t = & f_2(u_t, v_t, \xi(t))\,dt + \sigma(k) \circ dB_t, \end{cases}$$

where

$$\begin{cases} f_1(u,v,k) = h_1(k) + \dfrac{r(k)\alpha(k)(1-\eta(k))}{\alpha(k)+e^v} - \gamma(k)e^u - \dfrac{\beta(k)e^v}{1+a(k)e^u+b(k)e^v}, \\ f_2(u,v,k) = \dfrac{\theta(k)\beta(k)e^u}{1+a(k)e^u+b(k)e^v} - h_2(k) - \delta(k)e^v. \end{cases} \tag{3.1}$$

Fix a point $(u_0, v_0) \in \mathbb{E}$ and a smooth control function $\phi \in L^2([0,T];\mathbb{R})$, consider the following control system of integral equations

$$\begin{cases} u_\phi(t) = u_0 + \displaystyle\int_0^t (f_1(u_\phi(s), v_\phi(s)) + \sigma(k)\phi)\, ds, \\ v_\phi(t) = v_0 + \displaystyle\int_0^t (f_2(u_\phi(s), v_\phi(s)) + \sigma(k)\phi)\, ds. \end{cases} \tag{3.2}$$

Denote $\mathcal{D}_{u_0,v_0;\phi}$ be the Frechét derivative of the function $\mathfrak{f} \mapsto \mathbf{x}_{\phi+\mathfrak{f}}(T)$ from $L^2([0,T];\mathbb{R})$ to $\mathbb{R}^2_+$, where $\mathbf{x}_{\phi+\mathfrak{f}} = (u_{\phi+\mathfrak{f}}, v_{\phi+\mathfrak{f}})^T$. If for some $\phi \in L^2([0,T];\mathbb{R})$, the derivative $\mathcal{D}_{u_0,v_0;\phi}$ has rank 2, then $\mathcal{K}(T,u,v;u_0,v_0) > 0$ for $(u,v) = (u_\phi(T), v_\phi(T))$.

For $0 \le t_0 \le t \le T$, let $\mathcal{J}(t,t_0)$ be a matrix function such that $\mathcal{J}(t_0,t_0) = \mathbf{I}\mathrm{d}$, $\frac{\partial \mathcal{J}(t,t_0)}{\partial t} = \Psi(t)\mathcal{J}(t,t_0)$, where $\Psi = \mathbf{f}'(u_\phi, v_\phi)$, $\mathbf{f}'$ is the Jacobians of $\mathbf{f} = (f_1(u,v), f_2(u,v))^T$, and here $f_1(u,v,k)$ and $f_2(u,v,k)$ are defined in equation (3.1). Then

$$\mathcal{D}_{u_0,v_0;\phi}\mathfrak{f} = \int_0^T \mathcal{J}(T,s)\mathbf{g}(s)\mathfrak{f}(s)\, ds,$$

where $\mathbf{g} = (\sigma(k), \sigma(k))^T$.

Since we consider a continuous control function ϕ, the model (3.2) can be replaced by the following system of differential equations

$$\begin{cases} u'_\phi = h_1(k) + \dfrac{r(k)\alpha(k)(1-\eta(k))}{\alpha(k)+e^v} - \gamma(k)e^u - \dfrac{\beta(k)e^v}{1+a(k)e^u+b(k)e^v} + \sigma(k)\phi, \\ v'_\phi = \dfrac{\theta(k)\beta(k)e^u}{1+a(k)e^u+b(k)e^v} - h_2(k) - \delta(k)e^v + \sigma(k)\phi. \end{cases} \tag{3.3}$$

First, we claim that the rank of $\mathcal{D}_{u_0,v_0;\phi}$ is 2. Let $\epsilon \in (0,T)$ and $\mathfrak{f}(t) = \mathbf{1}_{[T-\epsilon,T]}$ for $t \in [0,T]$, where $\mathbf{1}_{[T-\epsilon,T]}$ is the characteristic function of interval $[T-\epsilon, T]$. By Taylor expansion, we obtain

$$\mathcal{J}(T,s) = \mathbf{I}\mathrm{d} + \Psi(T)(T-s) + o((T-s)).$$

Then

$$\mathcal{D}_{u_0,v_0;\phi}\mathfrak{f} = \epsilon\mathbf{g} + \frac{\epsilon^2}{2}\Psi(T)\mathbf{g} + o(\epsilon^2).$$

By calculating, we get

$$\Psi(T)\mathbf{g} = \begin{pmatrix} -\gamma(k)e^u + \dfrac{a(k)\beta(k)e^u e^v}{(1+a(k)e^u+b(k)e^v)^2} & -\dfrac{r(k)\alpha(k)(1-\eta(k))e^v}{(\alpha(k)+e^v)^2} - \dfrac{\beta(k)(1+a(k)e^u)e^v}{(1+a(k)e^u+b(k)e^v)^2} \\ \dfrac{\theta(k)\beta(k)e^u(1+b(k)e^v)}{(1+a(k)e^u+b(k)e^v)^2} & -\dfrac{\theta(k)\beta(k)b(k)e^u e^v}{(1+a(k)e^u+b(k)e^v)^2} - \delta(k)e^v \end{pmatrix}$$
$$\cdot \begin{pmatrix} \sigma(k) \\ \sigma(k) \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma(k)\sigma(k)e^u - \dfrac{r(k)\alpha(k)\sigma(k)(1-\eta(k))e^v}{(\alpha(k)+e^v)^2} - \dfrac{\beta(k)\sigma(k)e^v}{(1+a(k)e^u+b(k)e^v)^2} \\ \dfrac{\theta(k)\beta(k)\sigma(k)e^v}{(1+a(k)e^u+b(k)e^v)^2} - \sigma(k)\delta(k)e^v \end{pmatrix}.$$

Since $|\mathbf{g}\ \Psi(T)\mathbf{g}| \neq 0$, we get that $\mathbf{g}$ and $\Psi(T)\mathbf{g}$ are linearly independent, which imply that the derivative $\mathcal{D}_{u_0,v_0;\phi}$ is rank 2.

Second, we prove that for any two points $(u_0, v_0) \in \mathbb{E}$ and $(u_1, v_1) \in \mathbb{E}$, there exist a control function ϕ and $T > 0$ such that $(u_\phi(0), v_\phi(0)) = (u_0, v_0)$ and $(u_\phi(T), v_\phi(T)) = (u, v)$.

Let $w_\phi = v_\phi - \theta u_\phi$, model (3.3) becomes

$$\begin{cases} u'_\phi(t) = \mathbf{h}_1(u_\phi(t), w_\phi(t)) + \sigma(k)\phi, \\ w'_\phi(t) = \mathbf{h}_2(u_\phi(t), w_\phi(t)), \end{cases} \tag{3.4}$$

where

$$\begin{cases} \mathbf{h}_1(u_\phi(t), w_\phi(t)) = h_1(k) + \dfrac{r(k)\alpha(k)(1-\eta(k))}{\alpha(k)+e^{w_\phi(t)}e^{\theta(k)u_\phi(t)}} - \gamma(k)e^{u_\phi(t)} \\ \qquad\qquad - \dfrac{\beta(k)e^{w_\phi(t)}e^{\theta(k)u_\phi(t)}}{1+a(k)e^{u_\phi(t)}+b(k)e^{w_\phi(t)}e^{\theta(k)u_\phi(t)}}, \\ \mathbf{h}_2(u_\phi(t), w_\phi(t)) = -\theta(k)h_1(k) - h_2(k) - \dfrac{\theta(k)r(k)\alpha(k)(1-\eta(k))}{\alpha(k)+e^{w_\phi(t)}e^{\theta(k)u_\phi(t)}} + \theta(k)\gamma(k)e^{u_\phi(t)} \\ \qquad\qquad - \delta(k)e^{w_\phi(t)}e^{\theta(k)u_\phi(t)} + \dfrac{\theta(k)\beta(k)(e^{w_\phi(t)}e^{\theta(k)u_\phi(t)}+e^{u_\phi(t)})}{1+a(k)e^{u_\phi(t)}+b(k)e^{w_\phi(t)}e^{\theta(k)u_\phi(t)}}. \end{cases}$$

Now we prove for any $(u_0, w_0) \in \mathbb{E}$ and $(u_1, w_1) \in \mathbb{E}$, there exist a control function ϕ and $T > 0$ such that $(u_\phi(0), w_\phi(0)) = (u_0, w_0)$ and $(u_\phi(T), w_\phi(T)) = (u_1, w_1)$.

We construct the control function ϕ later. First, we find a positive constant T and a differentiable function $w_\phi : [0, T] \to \mathbb{E}$, such that $w_\phi(0) = w_0$, $w_\phi(T) = w_1$, $w'_\phi(0) = \mathbf{h}_2(u_0, w_0) = w_0^d$, $w'_\phi(T) = \mathbf{h}_2(u_1, w_1) = w_T^d$ and

$$w'_\phi(t) + \theta(k)h_1(k) + h_2(k) > 0, \qquad \textit{for } t \in [0, T]. \tag{3.5}$$

Second, dividing the structure of the function w_ϕ on three intervals $[0, \epsilon]$, $[\epsilon, T-\epsilon]$ and $[T-\epsilon, T]$, where $0 < \epsilon < \frac{T}{2}$.

It follows that we can construct a C^2-function $w_\phi : [0, \epsilon] \to \mathbb{E}$ such that

$$w_\phi(0) = w_0, \qquad w'_\phi(0) = w_0^d, \qquad w'_\phi(\epsilon) = 0$$

and w_ϕ satisfies (3.5) for $t \in [0, \epsilon]$. Analogously, we construct a C^2-function $w_\phi : [T-\epsilon, T] \to \mathbb{E}$ such that

$$w_\phi(T) = w_1, \qquad w'_\phi(T) = w_T^d, \qquad w'_\phi(T-\epsilon) = 0$$

and w_ϕ satisfies (3.5) for $t \in [T-\epsilon, T]$.

Taking T sufficiently large, we can extend the function

$$w_\phi : [0, \epsilon] \cap [T-\epsilon, T] \to \mathbb{E}$$

to a C^2-function w_ϕ defined on the whole interval $[0, T]$ such that the function z_ϕ satisfies (3.5) on $[0, T]$. It follows that we can find a C^1-function x_ϕ which satisfies the second equation of (3.4) and finally we can determine a continuous function ϕ from the first equation of (3.4).

Hence, we can prove that for any two points $(u_0, v_0) \in \mathbb{E}$ and $(u_1, v_1) \in \mathbb{E}$, there exist a control function ϕ and $T > 0$ such that $(u_\phi(0), v_\phi(0)) = (u_0, v_0)$ and $(u_\phi(T), v_\phi(T)) = (u, v)$.

Finally, based on the support theorems (see [1, 3, 26]), we obtain the positivity of $\mathcal{K}$, which implies that $\int_0^\infty \mathcal{T}_k(t) f\, dt > 0$, a.e. on $\mathbb{E}$. □

According to Lemma 3.2 and the Section 5 in [22] for any $t > 0$ the distribution of the process $(u(t), v(t), \xi(t))$ is absolutely continuous, then its density $\rho = (\rho_1, \rho_2, \cdots, \rho_N)^T$ satisfies the master equation

$$\frac{\partial \rho}{\partial t} = \Gamma' \rho + \mathcal{L}\rho, \tag{3.6}$$

where $\mathcal{L}\rho = (\mathcal{L}_1\rho_1, \mathcal{L}_2\rho_2, \cdots, \mathcal{L}_N\rho_N)^T$.

Let λ be a constant such that $\lambda > \max_{1\le k\le N}\{-\gamma_{kk}\}$ and $Q = \lambda^{-1}\Gamma' + \mathbf{I}$. Then equation (3.6) becomes

$$\frac{\partial \rho}{\partial t} = \lambda Q\rho - \lambda\rho + \mathcal{L}\rho. \tag{3.7}$$

Now, let $\mathbb{Y} = \mathbb{X} \times \mathbb{M}$, G be the σ-algebra of Borel subsets of $\mathbb{Y}$, and let $\tilde{m}$ be the product measure on $(\mathbb{Y}, G)$ given by $\tilde{m}(B \times \{k\}) = mB$ for each $B \in \Sigma$ and $k \in \mathbb{M}$. Obviously, $\mathcal{L}\rho$ generates a continuous Markov semigroup $\{\mathcal{T}(t)\}_{t\ge 0}$ on the space $L^1(\mathbb{Y}, G, \tilde{m})$, given by the formula

$$\mathcal{T}(t)g = (\mathcal{T}_1(t)g, \mathcal{T}_2(t)g, \cdots, \mathcal{T}_N(t)g),\ g(u, v, k) \in L^1(\mathbb{Y}, G, \tilde{m}).$$

It is obvious that the space $L^1(\mathbb{E}\times\mathbb{M})$ and $L^1(\mathbb{E})\times\cdots\times L^1(\mathbb{E})$ can be identified and therefore the semigroup $\mathcal{T}(t)$ is well-defined duo to $\mathbb{M}$ is a finite set.

From the Philips perturbation theorem[13], equation (3.7) with the initial condition $\rho(0, u, k) = f(u, k)$ generates a Markov semigroup $\{\mathcal{P}(t)\}_{t\ge 0}$ on the space $L^1(\mathbb{Y})$ given by

$$\mathcal{P}(t)f = e^{-\lambda t}\sum_{n=0}^{\infty} \lambda^n S^{(n)}(t)f,$$

where $S^{(0)}(t) = \mathcal{T}(t)$ and

$$S^{(n+1)}(t)f = \int_0^t S^{(0)}(t-s)QS^{(n)}(s)f\, ds, \qquad n \ge 0.$$

Step 3. Verify that the Markov semigroup satisfies the "Foguel alternative", that is, the semigroup $\{\mathcal{P}(t)\}_{t\ge 0}$ is asymptotically stable or sweeping with respect to compact sets, which can be seen in the following Lemma 3.4.

Lemma 3.4. *If $\theta(k)h_1(k) + h_2(k) > 0$, $k \in \mathbb{M}$. The semigroup $\{\mathcal{P}(t)\}_{t\ge 0}$ is asymptotically stable or sweeping with respect to compact sets.*

Proof. First, according to Lemma 3.3, and using the similar proof to Lemma 2.3 in [15] (or Lemma 4.4 in [12]), we can prove

$$\int_0^\infty \mathcal{P}(t)f\, dt > 0, \qquad \text{a.e. } on\ \mathbb{Y}.$$

Then, applying the similar arguments to Corollary 1 in [23], we obtain that for every $v_0 \in \mathbb{Y}$ there exist $\epsilon > 0$, $t > 0$ and a measurable function $\chi > 0$ such that

$$\int \chi\, d\tilde{m} > 0, \qquad \mathcal{K}(t, u, v) \ge \chi(u)$$

for $u \in \mathbb{Y}$ and $v \in \mathcal{B}(v_0, \epsilon)$, where $\mathcal{B}(v_0, \epsilon)$ is the open ball with center v_0 and radius r.

Finally, by virtue of Lemma 2.1, the proof of Lemma 3.4 is proved easily. □

Step 4. We prove that semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable.

According to Lemma 3.4, the semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable or sweeping with respect to compact sets. In order to prove that semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable, we need to construct a non-negative C^2-function V and $R > 0$ such that

$$\sup_{u^2+v^2>R^2} \mathscr{A}^* V(u, v, k) < 0,$$

where

$$\begin{aligned}\mathscr{A}^* V(u, v, k) =& \frac{\sigma^2(k)}{2}\Big(\frac{\partial^2 V}{\partial u^2} + 2\frac{\partial^2 V}{\partial u \partial v} + \frac{\partial^2 V}{\partial v^2}\Big) - f_1(u, v, k)\frac{\partial V}{\partial u} - f_2(u, v, k)\frac{\partial V}{\partial v}\\ &+ \sum_{k\neq l, k, l\in\mathbb{M}} \gamma_{kl}(V(u, v, l) - V(u, v, k)),\end{aligned}$$

where $f_1(u, v, k)$ and $f_2(u, v, k)$ are defined in equation (3.1).

Since the matrix Γ is irreducible, there exists a solution $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_N)$ of the Poisson system such that

$$\Gamma\varpi + \tilde{\kappa} = \sum_{k=1}^{N} \pi_k \kappa_k \mathbf{1},$$

where $\tilde{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_N)^T$ and $\mathbf{1} = (1, 1, \cdots, 1)^T$, that is, for any $k \in \mathbb{M}$, we have

$$\sum_{k\neq l, k, l\in\mathbb{M}} \gamma_{kl}(\varpi_l - \varpi_k) + \tilde{\kappa} = \sum_{k=1}^{N} \pi_k \kappa_k.$$

Constructing a C^2-function V:

$$V(u, v, k) =\Big(e^u - x^* - x^* \ln\frac{e^u}{x^*}\Big) + \frac{1 + a(k)x^*}{\theta(k)(1 + b(k)y^*)}\Big(e^v - y^* - y^* \ln\frac{e^v}{y^*}\Big) + (\varpi_k + |\varpi|),$$

where (x^*, y^*) is the positive equilibrium of model (1.1) without noise interference.

Then, by calculating, we have

$$\begin{aligned}\mathscr{A}^* V =& (e^u - x^*)\Big(h_1(k) + \frac{r(k)\alpha(k)(1 - \eta(k))}{\alpha(k) + e^v} - \gamma(k)e^u - \frac{\beta(k)e^v}{1 + a(k)e^u + b(k)e^v}\Big)\\ &+ \frac{1 + a(k)x^*}{\theta(k)(1 + b(k)y^*)}(e^v - y^*)\Big(\frac{\theta(k)\beta(k)e^u}{1 + a(k)e^u + b(k)e^v} - h_2(k) - \delta(k)e^v\Big)\\ &+ \frac{1}{2}\sigma^2(k)x^* + \frac{\sigma^2(k)(1 + a(k)x^*)y^*}{2\theta(k)(1 + b(k)y^*)} + \sum_{k\neq l, k, l\in\mathbb{M}} \gamma_{kl}(\varpi_l - \varpi_k)\\ =& - \gamma(k)(e^u - x^*)^2 + \frac{a(k)\beta(k)y^*(e^u - x^*)^2}{(1 + a(k)x^* + b(k)y^*)(1 + a(k)e^u + b(k)e^v)}\\ &- \frac{b(k)\beta(k)(1 + a(k)x^*)x^*(e^v - y^*)^2}{(1 + b(k)y^*)(1 + a(k)x^* + b(k)y^*)(1 + a(k)e^u + b(k)e^v)}\\ &+ \frac{r(k)\alpha(k)(1 - \eta(k))(e^u - x^*)(e^v - y^*)}{(\alpha(k) + y^*)(\alpha(k) + e^v)} + \frac{1}{2}\sigma^2(k)x^*\end{aligned}$$

$$
\begin{aligned}
&-\delta(k)(e^v-y^*)^2+\frac{\sigma^2(k)(1+a(k)x^*)y^*}{2\theta(k)(1+b(k)y^*)}+\sum_{k\neq l,k,l\in\mathbb{M}}\gamma_{kl}(\varpi_l-\varpi_k)\\
\leq&-\Big(\gamma(k)-\frac{a(k)\beta(k)y^*}{1+a(k)x^*+b(k)y^*}-\frac{r(k)(1-\eta(k))}{2\alpha(k)}\Big)(e^u-x^*)^2\\
&-\Big(\delta(k)-\frac{r(k)(1-\eta(k))}{2\alpha(k)}\Big)(e^v-y^*)^2+\sum_{k=1}^N\pi_k\kappa_k\\
\leq&-\Big(\hat{\gamma}-\frac{\check{a}\check{\beta}y^*}{1+\hat{a}x^*+\hat{b}y^*}-\frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)(e^u-x^*)^2\\
&-\Big(\hat{\delta}-\frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)(e^v-y^*)^2+\sum_{k=1}^N\pi_k\kappa_k.
\end{aligned}
$$

Under the conditions

$$
\sum_{k=1}^N\pi_k\kappa_k<\min\Big\{\Big(\hat{\gamma}-\frac{\check{a}\check{\beta}y^*}{1+\hat{a}x^*+\hat{b}y^*}-\frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)x^{*2},\Big(\hat{\delta}-\frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)y^{*2}\Big\}
$$

for $k\in\mathbb{M}$, we can imply that there exist $R\in\mathbb{R}^+$ large enough such that

$$
\sup_{u^2+v^2>R^2}\mathscr{A}^*V(u,v,k)\leq 0.
$$

Hence, by applying similar arguments to those in [18], the existence of a Khasminskiǐ function V implies that the semigroup is not sweeping from the ball $\{u^2+v^2\leq R^2\}\in\mathcal{B}(\mathbb{E})$, then, we obtain that semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable. It completes the proof of the Theorem 3.1 □

4 Extinction and Persistence

Consider the auxiliary equation

$$
\begin{cases} dX(t)=[(r(k)-\mu(k))X(t)-\gamma(k)X^2(t)]\,dt+\sigma(k)X(t)\,dB_1(t),\\ X(0)=x(0)>0.\end{cases}\tag{4.1}
$$

Using a method similar to that in Lemma 2.3 in [4], we can obtain the following Lemma 4.1.

Denote

$$
R_1(k)=r(k)-\mu(k)-\frac{\sigma^2(k)}{2},\qquad R_1^s=\sum_{k=1}^N\pi_kR_1(k).
$$

Lemma 4.1. *(i) When $R_1^s<0$, $X(t)$ is extinct, i.e.,* $\lim_{t\to+\infty}X(t)=0$ a.s.

(ii) When $R_1^s>0$, $X(t)$ is persistent in the mean and has a unique stationary distribution $\varrho(X,k)$.

Remark 4.2. Applying the comparison theorem of stochastic differential equation[11] results in

$$
x(t)\leq X(t),\qquad \text{a.s.}
$$

Denote

$$R_2(k) = \int_0^{+\infty} \frac{\theta(k)\beta(k)X}{1+a(k)X}\varrho(X,k)\mathrm{d}(X,k) - \Big(d(k) + \frac{\sigma^2(k)}{2}\Big),$$
$$\widetilde{R_2}(k) = R_2(k) - \frac{\theta(k)\beta(k)r(k)(1-\eta(k))}{\gamma(k)},$$

where $\varrho(X,k)$ is defined in Lemma 4.1.

Theorem 4.3. *(I) When $R_1^s < 0$, prey $x(t)$ is extinct, i.e.,*

$$\lim_{t\to+\infty} x(t) = 0, \qquad \text{a.s.}$$

(II) When $R_2^s = \sum_{k=1}^N \pi_k R_2(k) < 0$, predator $y(t)$ is extinct, i.e.,

$$\lim_{t\to+\infty} y(t) = 0, \qquad \text{a.s.}$$

(III) When $R_1^s > 0$ and $R_2^s < 0$, predator $y(t)$ is extinct and prey $x(t)$ is persistent in the mean, and has

$$\liminf_{t\to+\infty}\langle x(t)\rangle \geq \frac{R_1^s}{\check{\gamma}} > 0, \qquad \text{a.s.}$$

(IV) When $\widetilde{R_2}^s = \sum_{k=1}^N \pi_k \widetilde{R_2}(k) > 0$, predator $y(t)$ is persistent in the mean and has

$$\liminf_{t\to+\infty}\langle y(t)\rangle \geq \frac{\hat{a}\hat{\gamma}}{\check{b}\check{\theta}\check{\beta}\hat{\gamma} + \check{\theta}\check{\beta}^2\hat{a} + \check{\delta}\hat{a}\hat{\gamma}}\widetilde{R_2}^s > 0, \qquad \text{a.s.}$$

Proof. We can use the comparison theorem of stochastic differential equation and the results of Lemma 4.1 to prove (I–IV).

(I) Utilizing Itô's formula yields

$$\begin{aligned} d\ln x(t) =& \Big[r(\xi(t))\Big(\eta(\xi(t)) + \frac{\alpha(\xi(t))(1-\eta(\xi(t)))}{\alpha(\xi(t))+y(t)}\Big) - \mu(\xi(t)) - \gamma(\xi(t))x(t) \\ & - \frac{\beta(\xi(t))y(t)}{1+a(\xi(t))x(t)+b(\xi(t))y(t)} - \frac{\sigma^2(\xi(t))}{2}\Big]\,dt + \sigma(\xi(t))\,dB(t) \\ \leq& \Big[r(\xi(t)) - \mu(\xi(t)) - \frac{\sigma^2(\xi(t))}{2} - \gamma(\xi(t))x(t)\Big]\,dt + \sigma(\xi(t))\,dB(t). \end{aligned} \tag{4.2}$$

Integrating (4.2) from 0 to t and then dividing by t on both sides yields

$$\frac{\ln y(t) - \ln y(0)}{t} \leq \frac{1}{t}\int_0^t \Big(r(\xi(s)) - \mu(\xi(s)) - \frac{\sigma^2(\xi(s))}{2}\Big)\,ds + \frac{1}{t}\int_0^t \sigma(\xi(s))\,dB(s). \tag{4.3}$$

It follows from the ergodic property of $\xi(t)$ that

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t R_1(\xi(s))\,ds = \sum_{k=1}^N \pi_k R_1(k). \tag{4.4}$$

Taking the limit superior of both sides of (4.3) yields, and then applying the ergodic property of (4.4), one can calculate that

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \leq \sum_{k=1}^N \pi_k R_1(k) < 0, \qquad \text{a.s.,}$$

which implies

$$\lim_{t\to+\infty} x(t) = 0 \quad \text{a.s.}$$

(II) Utilizing Itô's formula gets

$$\begin{aligned}
d\ln y(t) =&\Big(\frac{\theta(\xi(t))\beta(\xi(t))x(t)}{1+a(\xi(t))x(t)+b(\xi(t))y(t)} - d(\xi(t)) - \delta(\xi(t))y(t) - \frac{\sigma^2(\xi(t))}{2}\Big)\,dt \\
&+ \sigma(\xi(t))\,dB(t) \\
=&\Big(\frac{\theta(\xi(t))\beta(\xi(t))X(t)}{1+a(\xi(t))X(t)} - \frac{\theta(\xi(t))\beta(\xi(t))X(t)}{1+a(\xi(t))X(t)} + \frac{\theta(\xi(t))\beta(\xi(t))x(t)}{1+a(\xi(t))x(t)+b(\xi(t))y(t)} \\
&- \delta(\xi(t))y(t) - d(\xi(t)) - \frac{\sigma^2(\xi(t))}{2}\Big)\,dt + \sigma(\xi(t))\,dB(t) \\
=&\Big(\frac{\theta(\xi(t))\beta(\xi(t))X(t)}{1+a(\xi(t))X(t)} - \frac{\theta(\xi(t))\beta(\xi(t))\big[b(\xi(t))X(t)y(t) + (X(t)-x(t))\big]}{(1+a(\xi(t))X(t))(1+a(\xi(t))x(t)+b(\xi(t))y(t))} \\
&- \delta(\xi(t))y(t) - d(\xi(t)) - \frac{\sigma^2(\xi(t))}{2}\Big)\,dt + \sigma(\xi(t))\,dB(t),
\end{aligned}$$

then, we have

$$d\ln y(t) \le \Big(\frac{\theta(\xi(t))\beta(\xi(t))X(t)}{1+a(\xi(t))X(t)} - d(\xi(t)) - \frac{\sigma^2(\xi(t))}{2}\Big)\,dt + \sigma(\xi(t))\,dB(t), \tag{4.5}$$

and

$$\begin{aligned}
d\ln y(t) \ge&\Big[\frac{\theta(\xi(t))\beta(\xi(t))X(t)}{1+a(\xi(t))X(t)} - \theta(\xi(t))\beta(\xi(t))(X(t)-x(t)) - d(\xi(t)) - \frac{\sigma^2(\xi(t))}{2} \\
&- \Big(\frac{b(\xi(t))\theta(\xi(t))\beta(\xi(t))}{a(\xi(t))} + \delta(\xi(t))\Big)y(t)\Big]\,dt + \sigma(\xi(t))\,dB(t).
\end{aligned} \tag{4.6}$$

Integrating (4.5) from 0 to t and then dividing by t on both sides yields

$$\begin{aligned}
\frac{\ln y(t) - \ln y(0)}{t} \le& \frac{1}{t}\int_0^t \frac{\theta(\xi(s))\beta(\xi(s))X(s)}{1+a(\xi(s))X(s)}\,ds - \frac{1}{t}\int_0^t \Big(d(\xi(s)) + \frac{\sigma^2(\xi(s))}{2}\Big)\,ds \\
&+ \frac{1}{t}\int_0^t \sigma(\xi(s))\,dB(s).
\end{aligned} \tag{4.7}$$

By the ergodic theorem, one gets

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\theta(\xi(s))\beta(\xi(s))X(s)}{1+a(\xi(s))X(s)}\,ds = \sum_{k=1}^N \int_0^\infty \frac{\theta(k)\beta(k)X}{1+a(k)X}\,\varrho(X,k)(\,dX,k), \qquad \text{a.s.}$$

It follows from the ergodic property of $\xi(t)$ that

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t R_2(\xi(s))\,ds = \sum_{k=1}^N \pi_k R_2(k). \tag{4.8}$$

Taking the limit superior of both sides of (4.7) yields, and then applying the ergodic property of (4.8), one can calculate that

$$\limsup_{t\to+\infty} \frac{\ln y(t)}{t} \le \sum_{k=1}^N \pi_k R_2(k) < 0, \qquad \text{a.s.,}$$

which implies

$$\lim_{t\to+\infty} y(t) = 0, \qquad \text{a.s.}$$

(III) Utilizing Itô's formula yields

$$\begin{aligned}\frac{\ln x(t) - \ln x(0)}{t} =& \frac{1}{t}\int_0^t \Big(r(\xi(s))\eta(\xi(s)) - \mu(\xi(s)) - \frac{\sigma^2(\xi(s))}{2}\Big)\, ds \\ &+ \frac{1}{t}\int_0^t \frac{r(\xi(s))\alpha(\xi(s))(1-\eta(\xi(s)))}{\alpha(\xi(s)) + y(s)}\, ds - \frac{1}{t}\int_0^t \gamma(\xi(s))x(s)\, ds \\ &- \frac{1}{t}\int_0^t \frac{\beta(\xi(s))y(s)}{1 + a(\xi(s))x(s) + b(\xi(s))y(s)}\, ds + \frac{1}{t}\int_0^t \sigma(\xi(s))\, dB(s). \end{aligned} \tag{4.9}$$

From the Theorem 4.3(II) and equation (4.9), for any $\epsilon > 0$ there is $T > 0$, when $t > T$,

$$\begin{aligned}\frac{\ln x(t) - \ln x(0)}{t} \geq& \frac{1}{t}\int_0^t \Big(r(\xi(s))\eta(\xi(s)) - \mu(\xi(s)) - \frac{\sigma^2(\xi(s))}{2}\Big)\, ds \\ &+ \frac{1}{t}\int_0^t \frac{r(\xi(s))\alpha(\xi(s))(1-\eta(\xi(s)))}{\alpha(\xi(s)) + \epsilon}\, ds - \frac{1}{t}\int_0^t \gamma(\xi(s))x(s)\, ds \\ &- \frac{1}{t}\int_0^t \beta(\xi(s))\epsilon\, ds + \frac{1}{t}\int_0^t \sigma(\xi(s))\, dB(s).\end{aligned}$$

Since ϵ is small enough, we have

$$\begin{aligned}&\frac{1}{t}\int_0^t \Big(r(\xi(s))\eta(\xi(s)) - \mu(\xi(s)) - \frac{\sigma^2(\xi(s))}{2}\Big)\, ds \\ &+ \frac{1}{t}\int_0^t \frac{r(\xi(s))\alpha(\xi(s))[1-\eta(\xi(s))]}{\alpha(\xi(s)) + \epsilon}\, ds - \frac{1}{t}\int_0^t \beta(\xi(s))\epsilon\, ds > 0.\end{aligned}$$

Thus

$$\liminf_{t\to+\infty} \langle x(t)\rangle \geq \frac{1}{\check{\gamma}}\sum_{k=1}^N \pi_k R_1(k) = \frac{R_1^s}{\check{\gamma}} > 0, \qquad \text{a.s.}$$

(IV) Applying Itô's formula to model (4.1), we obtain

$$\begin{aligned}\frac{\ln X(t) - \ln X(0)}{t} =& \frac{1}{t}\int_0^t \Big(r(\xi(s)) - \mu(\xi(s)) - \frac{\sigma^2(\xi(s))}{2}\Big)\, ds \\ &- \frac{1}{t}\int_0^t \gamma(\xi(s))X(s)\, ds + \frac{1}{t}\int_0^t \sigma(\xi(s))\, dB(s),\end{aligned}$$

and from equation (4.9)

$$\begin{aligned}\frac{\ln x(t) - \ln x(0)}{t} \geq& \frac{1}{t}\int_0^t \Big(r(\xi(s))\eta(\xi(s)) - \mu(\xi(s)) - \frac{\sigma^2(\xi(s))}{2}\Big)\, ds - \frac{1}{t}\int_0^t \gamma(\xi(s))x(s)\, ds \\ &- \frac{1}{t}\int_0^t \beta(\xi(s))y(s)\, ds + \frac{1}{t}\int_0^t \sigma(\xi(s))\, dB(s).\end{aligned}$$

Then, one can derive that

$$\frac{1}{t}\int_0^t \gamma(\xi(s))(x(s) - X(s))\, ds$$

$$\geq -\frac{1}{t}\int_0^t \beta(\xi(s))y(s)\,ds - \frac{1}{t}\int_0^t r(\xi(s))(1-\eta(\xi(s)))\,ds - \frac{\ln X(t)-\ln x(t)}{t}. \tag{4.10}$$

Then substituting (4.10) into (4.6) and combining Lemma 4.1, we have

$$\begin{aligned}\frac{\ln y(t)}{t} \geq& \frac{1}{t}\int_0^t \frac{\theta(\xi(s))\beta(\xi(s))X(s)}{1+a(\xi(s))X(s)}\,ds - \frac{\check{b}\check{\theta}\check{\beta}\hat{\gamma}+\check{\theta}\check{\beta}^2\hat{a}+\check{\delta}\hat{a}\hat{\gamma}}{\hat{a}\hat{\gamma}t}\int_0^t y(s)\,ds \\ &- \frac{1}{t}\int_0^t \Big(d(\xi(s))+\frac{\sigma^2(\xi(s))}{2} - \frac{\theta(\xi(s))\beta(\xi(s))r(\xi(s))(1-\eta(\xi(s)))}{\gamma(\xi(s))}\Big)\,ds \\ &+ \frac{1}{t}\int_0^t \sigma(\xi(s))\,dB(s) + \frac{\ln y(0)}{t}. \end{aligned} \tag{4.11}$$

Taking the limit superior of both sides of (4.11) yields

$$\liminf_{t\to+\infty}\langle y(t)\rangle \geq \frac{\hat{a}\hat{\gamma}}{\check{b}\check{\theta}\check{\beta}\hat{\gamma}+\check{\theta}\check{\beta}^2\hat{a}+\check{\delta}\hat{a}\hat{\gamma}}\sum_{k=1}^{N}\pi_k\widetilde{R_2}(k) > 0, \qquad \text{a.s.}$$

□

5 Conclusion and Numerical Simulation

In this paper, we use the Markov semigroup theory to investigate the stationary distribution of a novel stochastic hybrid predator-prey model with fear effect and Beddington-DeAngelis functional response. We have obtained the existence of asymptotically stable stationary distribution of the stochastic hybrid model, which can converge in L^1 to an invariant density under appropriate conditions in Theorem 3.1. Furthermore, the conditions for extinction and persistence of the stochastic hybrid model are also derived, as shown in Theorem 4.3. Specifically, it is shown that (i): the prey species will be extinct when $R_1^s < 0$, while prey species will be persistent in the mean when $R_1^s > 0$; (ii): the predator species will be extinct when $R_2^s < 0$, while predator species will be persistent in the mean when $\widetilde{R_2}^s > 0$.

Let $\xi(t)$ is a right-continuous Markov chain taking values in $\mathbb{M} = \{1, 2\}$ and the generator Γ of the Markov chain is

$$\Gamma = \begin{pmatrix} -0.7 & 0.7 \\ 0.3 & -0.3 \end{pmatrix}.$$

Then we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2) = (0.3, 0.7).$$

We choose the values of the parameters in model (1.1) as follows

$$\begin{aligned}
&r(1)=0.8, \quad \alpha(1)=0.8, \quad \beta(1)=0.6, \quad a(1)=0.1, \quad b(1)=0.15, \quad d(1)=0.1,\\
&\mu(1)=0.2, \quad \eta(1)=0.64, \quad \delta(1)=0.4, \quad \gamma(1)=0.5, \quad \theta(1)=0.8, \quad \sigma(1)=0.2,\\
&r(2)=0.75, \quad \alpha(2)=0.7, \quad \beta(2)=0.7, \quad a(2)=0.2, \quad b(2)=0.2, \quad d(2)=0.2,\\
&\mu(2)=0.2, \quad \eta(2)=0.82, \quad \delta(2)=0.34, \quad \gamma(2)=0.5, \quad \theta(2)=0.76, \quad \sigma(2)=0.1.
\end{aligned}$$

Note that

$$(x^*, y^*) = (0.5602, 0.4133), \qquad \hat{\gamma} = 0.5 > \frac{\check{a}\check{\beta}y^*}{1+\hat{a}x^*+\hat{b}y^*} + \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}} = 0.2575,$$

$$\hat{\delta} = 0.34 > \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}} = 0.2057,$$
$$\theta(1)h_1(1) + h_2(1) = 0.3536 > 0, \qquad \theta(2)h_1(2) + h_2(2) = 0.4786 > 0,$$
$$\sum_{k=1}^{N} \pi_k \kappa_k = 0.0086 < \min\Big\{\Big(\hat{\gamma} - \frac{\check{a}\check{\beta}y^*}{1+\hat{a}x^*+\hat{b}y^*} - \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)x^{*2}, \Big(\hat{\delta} - \frac{\check{r}(1-\hat{\eta})}{2\hat{\alpha}}\Big)y^{*2}\Big\}$$
$$= \min\{0.076, 0.0229\},$$

which means that the conditions of Theorem 3.1 are satisfied. Therefore, the model (1.1) has a stationary distribution and its asymptotically stable (see Fig.5.1).

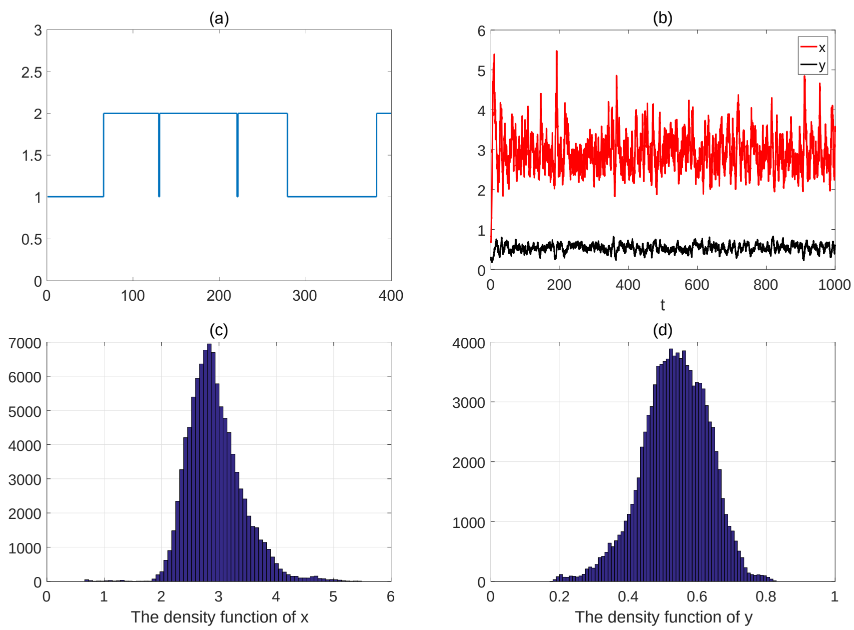

Fig. 5.1. (a): the Markovian chain of model (1.1). (b): the sample path of $x(t)$ and $y(t)$. (c): the smoothing curve of the probability density functions of $x(t)$. (d): the smoothing curve of the probability density functions of $y(t)$.

Conflict of Interest

The authors declare no conflict of interest.

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