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Stationary distribution of a stochastic reaction–diffusion predator–prey model with additional food and fear effect

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A B S T R A C T

Providing additional food supplements to predators is one of the crucial and effective tools for biological conservation and pest management, which has been widely recognized and validated by theoretical and experimental scientists. This paper focuses on the existence and uniqueness of stationary distribution of a stochastic reaction–diffusion predator–prey model that incorporates additional food for predator and fear effect for prey. It reveals that spatial diffusion is beneficial for maintaining the steady state of the model, whereas environment white noise has a negative effect.

1. Introduction

Understanding and controlling ecological systems and protecting them has long been a topic of active research for biologists and mathematicians. One promising approach for protecting certain species in ecological systems is to provide additional food to predators, which reduces predation and killings. Recently, scientists have demonstrated that this approach is widely recognized as one of the most important and effective tools for biological conservation and pest management [1–5]. This approach is primarily aimed at providing predators with alternative food supplements to reduce hunting of previously targeted species. However, the spatial diffusion of organisms and the environmental noise may affect the dynamic behaviors of prey and predator populations simultaneously in nature under some special scenarios. Therefore, it is of great research value to explore the joint effects of spatial diffusion and environmental noise on the dynamic behaviors of prey and predator. With the help of a bio-mathematical model, we propose the following stochastic reaction–diffusion predator–prey model with additional food for predator and fear for prey based on previous research [4,6]

$$\begin{cases} \mathrm{d}u(t,x) = \Bigg[d_u \Delta u(t,x) + r_0 u(t,x)\left(\eta + \dfrac{\alpha(1-\eta)}{\alpha + v(t,x)}\right) - au^2(t,x) \\ \qquad - \dfrac{\beta u(t,x)v(t,x)}{\epsilon + \delta\xi + u(t,x) + \rho v(t,x)} \Bigg] \mathrm{d}t + \sigma_1 u(t,x)\mathrm{d}W_1(t), \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\ \mathrm{d}v(t,x) = \left[d_v \Delta v(t,x) + \dfrac{c\beta(u(t,x)+\xi)v(t,x)}{\epsilon + \delta\xi + u(t,x) + \rho v(t,x)} - mv(t,x)\right]\mathrm{d}t + \sigma_2 v(t,x)\mathrm{d}W_2(t), \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\ \dfrac{\partial u(t,x)}{\partial \mathbf{n}} = \dfrac{\partial v(t,x)}{\partial \mathbf{n}} = 0, \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ u(0,x) = u_0(x) \geq 0, \; v(0,x) = v_0(x) \geq 0, \text{ in } \mathcal{O}, \end{cases} \tag{1}$$

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where u and v represent the biomass of prey and predator, respectively. r_0 stands for the intrinsic growth rate of prey. a stands for the decay rate due to intra-species competition. η stands the cost of minimum fear with $\eta \in [0,1]$ and α is the level of fear with prey. β is the rate of predation. c and m respectively are the conversion efficiency and mortality rate of predator. ξ is the quantity of the additional food supplied to the predator, and ρ is the strength of mutual interference among predators. ϵ is normalization coefficient relating the densities of prey and predator to the environment in which they interact. δ is the ratio between the handling times towards the additional food and the prey. $\sigma_i > 0$ $(i = 1,2)$ represent the intensities of the white noise. $d_u \geq 0$ and $d_v \geq 0$ represent the diffusion rate of prey and predator, respectively. Δ is the usual Laplacian operator in $\Omega \subset \mathbb{R}$, $\mathcal{O}$ is a bounded smooth domain of $\mathbb{R}^l (l \geq 1)$, $\mathbf{n}$ is the outward normal to $\partial\mathcal{O}$. There is no population flux across the boundaries due to the Neumann boundary conditions. $W_1(t,x)$ and $W_2(t,x)$ are $L^2(\mathcal{O},\mathbb{R})$-valued Wiener processes [7].

For model (1), we mainly investigate the existence and uniqueness of stationary distribution. In fact, there are few studies on the stationary distribution of the stochastic partial differential equations (SPDEs) models [8–10]. As we know, Liu [8] considered stationarity of a class of second-order stochastic evolution equations with memory, driven by Wiener processes or Lévy jump processes in Hilbert spaces, and Pan et al. [9] has applied this strategy to explore the existence and uniqueness of stationary distribution of a stochastic reaction–diffusion vegetation-water model disturbed by Ornstein–Uhlenbeck process. In this paper, we also apply the strategy in [8] to investigate the existence and uniqueness of stationary distribution of the model (1), which is presented in Section 2.

2. Stationary distribution

From [8], define

$$\mathcal{H} := \left\{ \varphi : \varphi \in L^2(\mathcal{O}),\ \frac{\partial \varphi}{\partial x_i} \in L^2(\mathcal{O}),\ i = 1,2 \right\},$$

and $\mathcal{H}^{-1}$ is the dual space of $\mathcal{H}$. Denote $\|\cdot\|$ is the norms in $\mathcal{H}$, and $\langle \cdot,\cdot \rangle$ is the duality product between $\mathcal{H}$ and $\mathcal{H}^{-1}$. Denote $\mathfrak{L}(\mathbb{H})$ to be the set of all bounded continuous real-valued functions over $\mathbb{H}$, where $\mathbb{H} = \mathcal{H} \times \mathcal{H}$. Let $\mathcal{B}(\mathbb{H})$ is the Borel algebra over $\mathbb{H}$, then $\mathcal{P}(\mathbb{H})$ is the space of all probability measures over $(\mathcal{O}, \mathcal{B}(\mathbb{H}))$.

Definition 2.1 (*[11]*). For all solutions $\hbar(t,x) = (u(t,x), v(t,x)) \in \mathbb{H}$ of the model (1), if exists a probability measure $\pi \in \mathcal{P}(\mathbb{H})$ such that

$$\pi(g) = \pi\left(\mathbb{P}_t g\right),$$

where

$$\pi(g) := \int_{\mathbb{H}} g(\xi)\pi(\mathrm{d}\xi) \text{ and } \mathbb{P}_t g(\xi) := \mathbb{E} g(\hbar(t,x,\xi)),\ g \in \mathfrak{L}(\mathbb{H}).$$

then a probability measure $\pi \in \mathcal{P}(\mathbb{H})$ is a stationary distribution.

Moreover, for $\pi_1, \pi_2 \in \mathcal{P}(\mathbb{H})$, define a metric on $\mathcal{P}(\mathbb{H})$ by

$$d(\pi_1,\pi_2) = \sup_{g\in\mathcal{M}} \left| \int_{\mathbb{H}} g(\xi)\pi_1(\mathrm{d}\xi) - \int_{\mathbb{H}} g(\zeta)\pi_2(\mathrm{d}\zeta) \right|,$$

where

$$\mathcal{M} := \left\{ g : \mathbb{H} \to \mathbb{R},\ |g(\xi) - g(\zeta)| \leq \|\xi - \zeta\|,\ \forall\ \phi, \psi \in \mathbb{H} \text{ and } |g(\cdot)| \leq 1 \right\}.$$

Hence, $\mathcal{P}(\mathbb{H})$ is complete under the metric $d(\cdot,\cdot)$.

Lemma 2.1. *There exists a positive constant K such that*

$$\int_{\mathcal{O}} \left(u(t,x) + \frac{1}{c} v(t,x) \right) \mathrm{d}x \leq K \text{ a.s.}$$

Proof. Define $\mathcal{V}(t,x) = \int_{\mathcal{O}} \left(u(t,x) + \frac{1}{c} v(t,x) \right) \mathrm{d}x$. Applying Itô's formula leads to

$$\begin{aligned}
\mathrm{d}\mathcal{V}(t,x) = \int_{\mathcal{O}} \Bigg[& r_0 u(t,x) \left(\eta + \frac{\alpha(1-\eta)}{\alpha + v(t,x)} \right) - a u^2(t,x) + d_u \Delta u(t,x) + \frac{d_v}{c} \Delta v(t,x) \\
& + \frac{\beta \xi v(t,x)}{\epsilon + \delta\xi + u(t,x) + \rho v(t,x)} - \frac{m}{c} v(t,x) \Bigg] \mathrm{d}x\mathrm{d}t + \int_{\mathcal{O}} \sigma_1 u(t,x)\mathrm{d}x\mathrm{d}W_1(t) + \int_{\mathcal{O}} \frac{\sigma_2}{c} v(t,x)\mathrm{d}x\mathrm{d}W_2(t) \\
\leq & \left(\frac{(r_0+m)^2}{4a} + \frac{\beta\xi}{\rho} \right) \mathrm{d}t - m \int_{\mathcal{O}} \left(u(t,x) + \frac{v(t,x)}{c} \right) \mathrm{d}x\mathrm{d}t + \int_{\mathcal{O}} \sigma_1 u(t,x)\mathrm{d}x\mathrm{d}W_1(t) + \int_{\mathcal{O}} \frac{\sigma_2}{c} v(t,x)\mathrm{d}x\mathrm{d}W_2(t) \\
\leq & \left[K_1 - m\mathcal{V}(t,x) \right] \mathrm{d}t + \int_{\mathcal{O}} \sigma_1 u(t,x)\mathrm{d}x\mathrm{d}W_1(t) + \int_{\mathcal{O}} \frac{\sigma_2}{c} v(t,x)\mathrm{d}x\mathrm{d}W_2(t),
\end{aligned}$$

where $K_1 = \frac{(r_0+m)^2}{4a} + \frac{\beta\xi}{\rho}$.

Next, we use the standard method in Lemma 2.3 in [12] to prove it, which is omitted here to avoid repetition. □

Lemma 2.2 ([8]). *Suppose that for any bounded subset $\mathbb{U} \subset \mathbb{H}$ and $\ell \geq 1$,*
(i) $\sup_{t\geq 0}\sup_{\xi\in\mathbb{U}} \mathbb{E}\|\hbar(t,x,\xi)\|^{\ell} < \infty$;
(ii) $\lim_{t\to\infty}\sup_{\xi,\zeta\in\mathbb{U}} \mathbb{E}\|\hbar(t,x,\xi)-\hbar(t,x,\zeta)\|^{\ell} = 0$.
Then the process $\hbar(t,x,\xi)$ has a stationary distribution.

To prove the existence and uniqueness of the stationary distribution of model (1), we only need to verify that (i) and (ii) of Lemma 2.2 are satisfied, that is, to prove the following Theorems 2.1 and 2.2.

Theorem 2.1. *For any $\ell > 1$, we have*

$$\sup_{0\leq t\leq T} \mathbb{E}\left(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}\right) \leq C_2,$$

where $C_2 = 2\left(\|u_0\|^{\ell} + \|v_0\|^{\ell}\right) e^{2C_1T}$ is a constant, where

$$C_1 = \max\left\{\ell r_0 + \frac{1}{2}\ell(\ell-1)\sigma_1^2 + 8\sqrt{2}\ell^2\sigma_1^2, \frac{\ell c\beta[\epsilon+(1+\delta)\xi]}{\epsilon+\delta\xi} + \frac{1}{2}\ell(\ell-1)\sigma_2^2 + 8\sqrt{2}\ell^2\sigma_2^2\right\}.$$

Proof. First, for $\ell \geq 1$, applying Itô's Lemma to $\|u_0\|^{\ell} + \|v_0\|^{\ell}$ yields

$$\begin{aligned}
&\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell} \\
=&\|u(0,x)\|^{\ell} + \|v(0,x)\|^{\ell} + \int_0^t \Bigg(\ell r_0\eta\|u(r,x)\|^{\ell} - \ell a\|u(r,x)\|^{1+\ell} - \ell m\|v(r,x)\|^{\ell} \\
&+ \ell\|u(r,x)\|^{\ell-2}\left\langle u(r,x), \frac{r_0\alpha(1-\eta)u(r,x)}{\alpha+v(r,x)}\right\rangle - \ell\|u(r,x)\|^{\ell-2}\left\langle u(r,x), \frac{\beta u(r,x)v(t,x)}{\epsilon+\delta\xi+u(r,x)+\rho v(r,x)}\right\rangle \\
&+ \ell\|v(r,x)\|^{\ell-2}\left\langle v(r,x), \frac{c\beta(u(r,x)+\xi)v(r,x)}{\epsilon+\delta\xi+u(r,x)+\rho v(r,x)}\right\rangle + \ell d_u\|u(r,x)\|^{\ell-2}\langle\Delta u(r,x), u(r,x)\rangle \\
&+ \ell d_v\|v(r,x)\|^{\ell-2}\langle\Delta v(r,x), v(r,x)\rangle + \frac{1}{2}\ell(\ell-1)\left(\sigma_1^2\|u(r,x)\|^{\ell} + \sigma_2^2\|v(r,x)\|^{\ell}\right)\Bigg)\mathrm{d}r \\
&+ \int_0^t \ell\sigma_1\|u(r,x)\|^{\ell}\mathrm{d}W_1(r) + \int_0^t \ell\sigma_2\|v(r,x)\|^{\ell}\mathrm{d}W_2(r).
\end{aligned} \tag{2}$$

Taking the expectation and $\sup(\cdot)$ to (2), and then using Young inequality and Burkholder–Davis–Gundy inequality, one has

$$\begin{aligned}
&\sup_{0\leq t\leq T}\mathbb{E}(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}) \\
\leq&\mathbb{E}(\|u(0,x)\|^{\ell} + \|v(0,x)\|^{\ell}) + \sup_{0\leq t\leq T}\mathbb{E}\int_0^t \Bigg(\ell r_0\|u(r,x)\|^{\ell} - \ell a\|u(r,x)\|^{1+\ell} + \frac{\ell c\beta[\epsilon+(1+\delta)\xi]}{\epsilon+\delta\xi}\|v(r,x)\|^{\ell} \\
&- \ell d_u\|u(r,x)\|^{\ell-2}\|\nabla x(r,x)\|^2 - \ell d_v\|v(r,x)\|^{\ell-2}\|\nabla v(r,x)\|^2 + \frac{1}{2}\ell(\ell-1)\left(\sigma_1^2\|u(r,x)\|^{\ell} + \sigma_2^2\|v(r,x)\|^{\ell}\right)\Bigg)\mathrm{d}r \\
&+ \sup_{0\leq t\leq T}\mathbb{E}\left|\int_0^t \ell\sigma_1\|u(r,x)\|^{\ell}\mathrm{d}W_1(r)\right| + \sup_{0\leq t\leq T}\mathbb{E}\left|\int_0^t \ell\sigma_2\|v(r,x)\|^{\ell}\mathrm{d}W_2(r)\right| \\
\leq&\mathbb{E}\left(\|u(0,x)\|^{\ell} + \|v(0,x)\|^{\ell}\right) + \sup_{0\leq t\leq T}\mathbb{E}\int_0^t \left(\ell r_0 + \frac{1}{2}\ell(\ell-1)\sigma_1^2\right)\|u(r,x)\|^{\ell}\mathrm{d}r \\
&+ \sup_{0\leq t\leq T}\mathbb{E}\int_0^t \left(\frac{\ell c\beta[\epsilon+(1+\delta)\xi]}{\epsilon+\delta\xi} + \frac{1}{2}\ell(\ell-1)\sigma_2^2\right)\|v(r,x)\|^{\ell}\mathrm{d}r \\
&+ 4\sqrt{2}\sup_{0\leq t\leq T}\mathbb{E}\|u(t,x)\|^{\frac{\ell}{2}}\left[\int_0^t \ell^2\sigma_1^2\|u(r,x)\|^{\ell}\mathrm{d}r\right]^{\frac{1}{2}} + 4\sqrt{2}\sup_{0\leq t\leq T}\mathbb{E}\|v(t,x)\|^{\frac{\ell}{2}}\left[\int_0^t \ell^2\sigma_2^2\|v(r,x)\|^{\ell}\mathrm{d}r\right]^{\frac{1}{2}} \\
\leq&\mathbb{E}\left(\|u(0,x)\|^{\ell} + \|v(0,x)\|^{\ell}\right) + \sup_{0\leq t\leq T}\mathbb{E}\int_0^t \left(\ell r_0 + \frac{1}{2}\ell(\ell-1)\sigma_1^2\right)\|u(r,x)\|^{\ell}\mathrm{d}r + \frac{1}{2}\sup_{0\leq t\leq T}\mathbb{E}\left(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}\right) \\
&+ \sup_{0\leq t\leq T}\mathbb{E}\int_0^t \left(\frac{\ell c\beta[\epsilon+(1+\delta)\xi]}{\epsilon+\delta\xi} + \frac{1}{2}\ell(\ell-1)\sigma_2^2\right)\|v(r,x)\|^{\ell}\mathrm{d}r + 8\sqrt{2}\sup_{0\leq t\leq T}\mathbb{E}\int_0^t \ell^2\left(\sigma_1^2\|u(r,x)\|^{\ell} + \sigma_2^2\|v(r,x)\|^{\ell}\right)\mathrm{d}r \\
\leq&\mathbb{E}\left(\|u(0,x)\|^{\ell} + \|v(0,x)\|^{\ell}\right) + C_2\sup_{0\leq t\leq T}\mathbb{E}\int_0^t \left(\|u(r,x)\|^{\ell} + \|v(r,x)\|^{\ell}\right)\mathrm{d}r + \frac{1}{2}\sup_{0\leq t\leq T}\mathbb{E}\left(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}\right).
\end{aligned}$$

Hence,

$$\sup_{0\leq t\leq T}\mathbb{E}\left(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}\right) \leq 2\mathbb{E}\left(\|u_0\|^{\ell} + \|v_0\|^{\ell}\right) + 2C_2\sup_{0\leq t\leq T}\mathbb{E}\int_0^t \left(\|u(r,x)\|^{\ell} + \|v(r,x)\|^{\ell}\right)\mathrm{d}r.$$

By applying the Gronwall's inequality, one gets

$$\sup_{0\leq t\leq T}\mathbb{E}\left(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}\right) \leq 2\left(\|u_0\|^{\ell} + \|v_0\|^{\ell}\right)e^{2C_1T} := C_2.$$

The proof is completed. □

Theorem 2.2. *Suppose that there exists a constant $\ell > 1$ such that* $\max\{I_1, I_2\} < 0$, *where*

$$\begin{aligned} I_1 =& \ell\left(\frac{r_0(\alpha+(1-\eta)K)}{\alpha}+\beta+\frac{\beta}{\rho}\right)-\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)-\ell\mathfrak{I}d_u+2\ell aK+\frac{c\beta(\epsilon+(\delta+\rho)\xi)}{\rho(\epsilon+\delta\xi)}+\frac{1}{2}\ell(\ell-1)\sigma_1^2,\\ I_2 =& \frac{r_0(1-\eta)K}{\alpha}+\beta+\frac{c\beta(\ell(1+\rho)-1)(\epsilon+(\delta+\rho)\xi)}{\rho(\epsilon+\delta\xi)}-\ell\mathfrak{I}d_v-\ell m+\frac{1}{2}\ell(\ell-1)\sigma_2^2, \end{aligned}$$

with

$$\mathfrak{I}=\min\left\{\inf_{\imath\in\mathbb{H}}\frac{\|\nabla\imath(t,x,\phi,\psi)\|^2}{\|\imath(t,x,\phi,\psi)\|^2},\inf_{\jmath\in\mathbb{H}}\frac{\|\nabla\jmath(t,x,\phi,\psi)\|^2}{\|\jmath(t,x,\phi,\psi)\|^2}\right\}.$$

Then the model (1) has a unique stationary distribution $\pi\in\mathcal{P}(\mathbb{H})$.

Proof. For all $\ell > 1$, $\lambda > 0$, denote

$$\begin{aligned} \Theta(t,x,\phi,\psi) =& e^{\lambda t}\left(\|u(t,x,\phi)-u(t,x,\psi)\|^{\ell}+\|v(t,x,\phi)-v(t,x,\psi)\|^{\ell}\right)\\ =& e^{\lambda t}\left(\|\imath(t,x,\phi,\psi)\|^{\ell}+\|\jmath(t,x,\phi,\psi)\|^{\ell}\right). \end{aligned}$$

By using Itô's Lemma yields

$$\begin{aligned} \mathrm{d}\Theta(t,x,\phi,\psi) =& \lambda\Theta(t,x,\phi,\psi)\mathrm{d}t+e^{\lambda t}\Bigg[\ell\|\imath(t,x,\phi,\psi)\|^{\ell-2}\langle\imath(t,x,\phi,\psi),r_0u(t,x,\phi)\left(\eta+\frac{\alpha(1-\eta)}{\alpha+v(t,x,\phi)}\right)\\ &-r_0x(t,x,\psi)\left(\eta+\frac{\alpha(1-\eta)}{\alpha+v(t,x,\psi)}\right)-\frac{\beta u(t,x,\phi)v(t,x,\phi)}{\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi)}\\ &+\frac{\beta u(t,x,\psi)v(t,x,\psi)}{\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi)}\rangle-\ell a\|\imath(t,x,\phi,\psi)\|^{\ell}(\|u(t,x,\phi)+u(t,x,\psi)\|)\\ &+\ell\|\jmath(t,x,\phi,\psi)\|^{\ell-2}\langle\jmath(t,x,\phi,\psi),\frac{c\beta(u(t,x,\phi)+\xi)v(t,x,\phi)}{\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi)}\\ &-\frac{c\beta(u(t,x,\psi)+\xi)v(t,x,\psi)}{\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi)}\rangle-\ell m\|\jmath(t,x,\phi,\psi)\|^{\ell}\\ &+\ell d_u\|\imath(t,x,\phi,\psi)\|^{\ell-2}\langle\Delta\imath(t,x,\phi,\psi),\imath(t,x,\phi,\psi)\rangle+\frac{1}{2}\ell(\ell-1)\sigma_1^2\|\imath(t,x,\phi,\psi)\|^{\ell}\\ &+\ell d_v\|\jmath(t,x,\phi,\psi)\|^{\ell-2}\langle\Delta\jmath(t,x,\phi,\psi),\jmath(t,x,\phi,\psi)\rangle+\frac{1}{2}\ell(\ell-1)\sigma_2^2\|\jmath(t,x,\phi,\psi)\|^{\ell}\Bigg]\mathrm{d}t\\ &-\ell\sigma_1\|\imath(t,x,\phi,\psi)\|^{\ell}e^{\lambda t}\mathrm{d}W_1(t)-\ell\sigma_2\|\jmath(t,x,\phi,\psi)\|^{\ell}e^{\lambda t}\mathrm{d}W_2(t)\\ \le& \lambda\Theta(t,x,\phi,\psi)\mathrm{d}t+e^{\lambda t}\Bigg[\ell\|\imath(t,x,\phi,\psi)\|^{\ell-2}\langle\imath(t,x,\phi,\psi),P_1\rangle\\ &-\ell a\|\imath(t,x,\phi,\psi)\|^{\ell}(\|u(t,x,\phi)+u(t,x,\psi)\|)+\ell\|\jmath(t,x,\phi,\psi)\|^{\ell-2}\langle\jmath(t,x,\phi,\psi),P_2\rangle\\ &-\ell d_u\|\imath(t,x,\phi,\psi)\|^{\ell-2}\|\nabla\imath(t,x,\phi,\psi)\|^2-\ell d_v\|\jmath(t,x,\phi,\psi)\|^{\ell-2}\|\nabla\jmath(t,x,\phi,\psi)\|^2\\ &-\ell m\|\jmath(t,x,\phi,\psi)\|^{\ell}+\frac{1}{2}\ell(\ell-1)\sigma_1^2\|\imath(t,x,\phi,\psi)\|^{\ell}+\frac{1}{2}\ell(\ell-1)\sigma_2^2\|\jmath(t,x,\phi,\psi)\|^{\ell}\Bigg]\mathrm{d}t\\ &-\ell\sigma_1\|\imath(t,x,\phi,\psi)\|^{\ell}e^{\lambda t}\mathrm{d}W_1(t)-\ell\sigma_2\|\jmath(t,x,\phi,\psi)\|^{\ell}e^{\lambda t}\mathrm{d}W_2(t), \end{aligned} \tag{3}$$

where

$$\begin{aligned} P_1 =& r_0u(t,x,\phi)\left(\eta+\frac{\alpha(1-\eta)}{\alpha+v(t,x,\phi)}\right)-r_0x(t,x,\psi)\left(\eta+\frac{\alpha(1-\eta)}{\alpha+v(t,x,\psi)}\right)\\ &-\frac{\beta u(t,x,\phi)v(t,x,\phi)}{\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi)}+\frac{\beta u(t,x,\psi)v(t,x,\psi)}{\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi)},\\ P_2 =& \frac{c\beta(u(t,x,\phi)+\xi)v(t,x,\phi)}{\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi)}-\frac{c\beta(u(t,x,\psi)+\xi)v(t,x,\psi)}{\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi)}. \end{aligned}$$

Note that

$$\begin{aligned} &\ell\|\imath(t,x,\phi,\psi)\|^{\ell-2}\left\langle\imath(t,x,\phi,\psi),P_1\right\rangle\\ \le& \ell\|\imath(t,x,\phi,\psi)\|^{\ell-2}\Bigg\langle\imath(t,x,\phi,\psi),r_0\left(\eta+\frac{\alpha(1-\eta)}{\alpha+v(t,x,\phi)}\right)\imath(t,x,\phi,\psi)-\frac{r_0\alpha(1-\eta)u(t,x,\psi)\jmath(t,x,\phi,\psi)}{(\alpha+v(t,x,\phi))(\alpha+v(t,x,\psi))}\\ &-\frac{\beta[\epsilon+\delta\xi+\rho v(t,x,\psi)]v(t,x,\phi)\imath(t,x,\phi,\psi)+\beta[\epsilon+\delta\xi+u(t,x,\phi)]u(t,x,\psi)\jmath(t,x,\phi,\psi)}{(\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi))(\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi))}\Bigg\rangle\\ \le& \ell\left(r_0+\frac{\beta}{\rho}\right)\|\imath(t,x,\phi,\psi)\|^{\ell}+\ell\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)\|\imath(t,x,\phi,\psi)\|^{\ell-1}\|\jmath(t,x,\phi,\psi)\|. \end{aligned}$$

Then, using Young inequality leads to

$$\begin{aligned}&\ell\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)\|\imath(t,x,\phi,\psi)\|^{\ell-1}\|\jmath(t,x,\phi,\psi)\|\\ \leq&(\ell-1)\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)\|\imath(t,x,\phi,\psi)\|^{\ell}+\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)\|\jmath(t,x,\phi,\psi)\|^{\ell}.\end{aligned}$$

Thus

$$\begin{aligned}\ell\|\imath(t,x,\phi,\psi)\|^{\ell-2}\Big\langle\imath(t,x,\phi,\psi),P_1\Big\rangle\leq&\left[\ell\left(\frac{r_0(\alpha+(1-\eta)K)}{\alpha}+\beta+\frac{\beta}{\rho}\right)-\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)\right]\|\imath(t,x,\phi,\psi)\|^{\ell}\\&+\left(\frac{r_0(1-\eta)K}{\alpha}+\beta\right)\|\jmath(t,x,\phi,\psi)\|^{\ell}.\end{aligned}\tag{4}$$

Similarly,

$$\begin{aligned}&\ell\|\jmath(t,x,\phi,\psi)\|^{\ell-2}\Big\langle\jmath(t,x,\phi,\psi),P_2\Big\rangle\\ \leq&\ell\|\jmath(t,x,\phi,\psi)\|^{\ell-2}\Big\langle\jmath(t,x,\phi,\psi),\frac{c\beta\rho\xi[[\epsilon+\delta\xi+u(t,x,\psi)]\jmath(t,x,\phi,\psi)-v(t,x,\psi)\imath(t,x,\phi,\psi)]}{(\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi))(\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi))}\\&+\frac{c\beta[\epsilon+\delta\xi+\rho v(t,x,\psi)]v(t,x,\phi)\imath(t,x,\phi,\psi)+c\beta[\epsilon+\delta\xi+u(t,x,\phi)]u(t,x,\psi)\jmath(t,x,\phi,\psi)}{(\epsilon+\delta\xi+u(t,x,\phi)+\rho v(t,x,\phi))(\epsilon+\delta\xi+u(t,x,\psi)+\rho v(t,x,\psi))}\Big\rangle\\ \leq&\frac{c\beta(\epsilon+(\delta+\rho)\xi)}{\rho(\epsilon+\delta\xi)}\|\imath(t,x,\phi,\psi)\|^{\ell}+\frac{c\beta(\ell(1+\rho)-1)(\epsilon+(\delta+\rho)\xi)}{\rho(\epsilon+\delta\xi)}\|\jmath(t,x,\phi,\psi)\|^{\ell}.\end{aligned}\tag{5}$$

Substituting (4) and (5) into (3), and then integrating both sides and taking expectations, one gets

$$\begin{aligned}\mathbb{E}\Theta(t,x,\phi,\psi)\leq&\mathbb{E}\Theta(0,x,\phi,\psi)+\lambda\mathbb{E}\int_0^t\Theta(r,x,\phi,\psi)\mathrm{d}r\\&+I_1\mathbb{E}\int_0^t e^{\lambda r}\|\imath(r,x,\phi,\psi)\|^{\ell}\mathrm{d}r+I_2\mathbb{E}\int_0^t e^{\lambda r}\|\jmath(r,x,\phi,\psi)\|^{\ell}\mathrm{d}r.\end{aligned}\tag{6}$$

Taking the $\sup_{\phi,\psi\in\mathbb{U}}$ on (6) results in

$$\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\Theta(t,x,\phi,\psi)\leq\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\Theta(0,x,\phi,\psi)+(\lambda+C_3)\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\int_0^t\Theta(r,x,\phi,\psi)\mathrm{d}r,$$

where $C_3=\max\left\{I_1,I_2\right\}$.

Using the Gronwall's inequality leads to

$$\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\Theta(t,x,\phi,\psi)\leq\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\Theta(0,x,\phi,\psi)e^{(\lambda+C_3)t}.$$

That is,

$$\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\left(\|\imath(t,x,\phi,\psi)\|^{\ell}+\|\jmath(t,x,\phi,\psi)\|^{\ell}\right)\leq\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\Theta(0,x,\phi,\psi)e^{C_4t}.$$

Assumption $C_4<0$, then when $t\to\infty$, one has

$$\lim_{t\to\infty}\sup_{\phi,\psi\in\mathbb{U}}\mathbb{E}\left(\|\imath(t,x,\phi,\psi)\|^{\ell}+\|\jmath(t,x,\phi,\psi)\|^{\ell}\right)=0.$$

Thus, there has a stationary distribution $\pi\in\mathcal{P}(\mathbb{H})$.

Next, we certify the uniqueness. Assume that $\pi'\in\mathcal{P}(\mathbb{H})$ is another stationary distribution of the model (1). By a straightforward calculation, we obtain

$$|\pi(g)-\pi'(g)|\leq\int_{\mathbb{H}\times\mathbb{H}}|\mathbb{P}_tg(\phi)-\mathbb{P}_tg(\psi)|\pi(\mathrm{d}\phi)\pi'(\mathrm{d}\psi)\leq C_5e^{-\lambda t},$$

where $C_5>0$.

Thus

$$\lim_{t\to\infty}|\pi(g)-\pi'(g)|=0.$$

Therefore, we can deduce that the model (1) has a unique stationary distribution. □

3. Conclusion

This paper investigates the conditions for the existence and uniqueness of stationary distribution of a stochastic reaction–diffusion predator–prey model with additional food for predator and fear effect for prey according to a suitable Lyapunov functionals. In maintaining a stable coexistence state of populations in the model, spatial diffusion is a positive impact, while environment white noise plays a inhibiting role in the steady state and can destroy it. The research results can provide theoretical guidance for solving practical biological control problems, such as biological conservation and pest management.

Data availability

No data was used for the research described in the article.

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References

[1] P.D.N. Srinivasu, B.S.R.V. Prasad, M. Venkatesulu, Biological control through provision of additional food to predators: a theoretical study, Theor. Popul. Biol. 72 (1) (2007) 111–120.
[2] P.D.N. Srinivasu, B.S.R.V. Prasad, Time optimal control of an additional food provided predator–prey system with applications to pest management and biological conservation, J. Math. Biol. 60 (4) (2010) 591–613.
[3] P.D.N. Srinivasu, B.S.R.V. Prasad, Role of quantity of additional food to predators as a control in predator–prey systems with relevance to pest management and biological conservation, Bull. Math. Biol. 73 (10) (2011) 2249–2276.
[4] B.S.R.V. Prasad, M. Banerjee, P.D.N. Srinivasu, Dynamics of additional food provided predator-prey system with mutually interfering predators, Math. Biosci. 246 (1) (2013) 176–190.
[5] P.D.N. Srinivasu, D.K.K. Vamsi, V.S. Ananth, Additional food supplements as a tool for biological conservation of predator-prey systems involving type III functional response: A qualitative and quantitative investigation, J. Theoret. Biol. 455 (2018) 303–318.
[6] H. Qi, X. Meng, Threshold behavior of a stochastic predator-prey system with prey refuge and fear effect, Appl. Math. Lett. 113 (2021) 106846.
[7] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 2014.
[8] K. Liu, Stationary distributions of second order stochastic evolution equations with memory in Hilbert spaces, Stochastic Process. Appl. 130 (1) (2020) 366–393.
[9] S. Pan, Q. Zhang, A. Meyer-Baese, Stationary distribution of a stochastic vegetation–water system with reaction-diffusion, Appl. Math. Lett. 123 (2022) 107589.
[10] H. Qi, X. Meng, Global dynamics of a stochastic reaction-diffusion predator-prey system with space-time white noise, Internat. J. Systems Sci. (2023) 1–27.
[11] K.R. Parthasarathy, Probability Measures on Metric Spaces, Vol. 352, American Mathematical Soc., 2005.
[12] S. Zhao, S. Yuan, H. Wang, Threshold behavior in a stochastic algal growth model with stoichiometric constraints and seasonal variation, J. Differential Equations 268 (9) (2020) 5113–5139.